Higher Artin Stacks and Moduli Stacks

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To the first mathematician to write
“The proof is trivial.”
You were wrong.

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Abstract

It is self-evident that in general, given a class of spaces A with some equivalence relation $\sim$ of symmetries, we can expect its elements to vary in a geometric manner. One way to formalize this is to parameterize each element of $A/\sim$ by a point in some other space M, termed a *moduli space*. Unfortunately, should our spaces have non-trivial automorphisms, the relation $\sim$ may be too convoluted for a moduli space to exist. A solution is instead to encode geometric structure in our original moduli problem, producing something akin to a sheaf which remembers symmetries. This yields a *stack*, which when considered in the context of algebraic geometry and given further geometric properties becomes an *Artin stack*, a construction rich enough to yield interesting geometric data but general enough to exist in a vast range of cases.

However, sometimes even this is insufficient: we may be interested not only in some spaces and their automorphisms, but also the higher automorphisms of these, *ad infinitum*. Furthermore, the relation $\sim$ we choose may be induced by morphisms without inverses, such as quasi-isomorphisms or homotopy equivalences. In these cases, a notion of higher morphism is necessary, demanding we turn to higher category theory for aid. Such a line of inquiry leads us to Toën and Vezzosi's notion of a *higher Artin stack* [57, pg. 14], the correct setting for the problems posed above.

It is the goal of this dissertation to present the theory of higher Artin stacks. We begin with an overview of moduli spaces and the intertwined study of geometric invariant theory. We then proceed to classical stack theory, developing the standard notions of Grothendieck topology, fibred categories and descent data required. After this, we dive into higher category theory, exploring simplicial and Segal categories, model categories and localizations. Finally, we define a Segal category of higher stacks and higher Artin stacks, closing with some useful constructions like an étale cohomology for higher Artin stacks and some examples like stacks of abelian categories and stacks of perfect complexes.

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Chapter 1

Classical Moduli Spaces

> *"If there is one thing in mathematics that fascinates me more than anything else (and doubtless always has), it is neither "number" nor "size", but always form. And among the thousand-and-one faces whereby form chooses to reveal itself to us, the one that fascinates me more than any other and continues to fascinate me, is the structure hidden in mathematical things."*
>
> – Alexander Grothendieck

Our journey begins with the observation that a class or collection of spaces is seldom fully described by a mere set. As an example, consider the set A of all circles in $\mathbb{R}^3$, up to rotation and translation. A on its own seems ignorant of how its elements implicitly relate to one another geometrically; different circles could be compared, with some being 'very similar' and others 'very different' based on their radii. We should reasonably expect to be able to take limits of circles, or perhaps 'curves' varying smoothly through them.

Intrigued by this, our major question becomes the following:

> *Can we construct a mathematical object that makes explicit the geometric variations in a collection of spaces?*

This could be done in the above case by bijectively assigning each point on the positive real line $x \in \mathbb{R}_{>0}$ the unique circle $f(x) \in A$ with radius x. We can now concretely see the implicit geometry between the circles; as we move continuously through the real line, we also continuously vary through elements of A, letting us consider curves and limits in A by looking at $\mathbb{R}_{>0}$.

In some sense, $\mathbb{R}_{>0}$ parameterizes the elements of A. Using *moduli* to mean 'parameter', we deem $\mathbb{R}_{>0}$ to be a *moduli space* solving the *moduli problem* posed by A. It is in general through parameterizing a collection of spaces A by points on some such moduli space that we should expect to understand the geometric interrelations between them. The picture is usually more complex than this, including an equivalence relation $\sim$ on A that captures isomorphisms between spaces we would like to remember. For instance, our original problem could have been stated by letting A be the set of all circles in $\mathbb{R}^2$, with $\sim$ generated by the translation action of $\mathbb{R}^2$ on itself. Then our moduli space $\mathbb{R}_{>0}$ parameterizes $A/\sim$.

The reader should perhaps maintain a healthy dose of caution that our question said ‘mathematical object’ rather than ‘space’; indeed, moduli spaces will not be the only solution we ultimately consider to moduli problems.

1.1 Fine and Coarse Moduli Spaces

The above example is formally a case of a *fine moduli space*, which we define as in [39, ch. 1]. To do so, we must first define a family parameterized by a ‘space’, or an object of the category we wish to work within. We will use the category $\mathbf{Var}_k$ of k-varieties for some fixed field k; our definition of a variety follows that of [50], but this is really not so important.

For a set of objects A with an equivalence relation $\sim$, a family parameterized by some $S \in \mathbf{Var}_k$ will be, in essence, an assignment of an element of A to each point on S that is suitably geometric. For such a family E and $s \in S$, we write $E_s \in A$ for the element of A assigned by E to s. We must also define a way to extend $\sim$ to families parameterized by S, so for families E, F we have $E \sim F$ precisely when $E_s \sim F_s$ for all $s \in S$.

As an example from [39, pg. 16], for some fixed variety X, let A be the set of all vector bundles over X as defined in [39, pg. 9] and $\sim$ isomorphism of vector bundles. For $S \in \mathbf{Var}_k$, a family E parameterized by S will be chosen to be a vector bundle over $S \times X$. We can then declare $E \sim F$ for families E, F parameterized by S when they are isomorphic as vector bundles. It is noted in [39] that this $\sim$ is defined somewhat naively, but we leave this simply as a motivating example.

Another example from [39, pg. 23] in the set $\mathbf{End}_n$ of all n-dimensional linear endomorphisms $V \to V$ for all vector spaces V and fixed n. Here, our equivalence relation $\sim$ is similarity. We may choose a family E parameterized by S to now be a pair (V, T), where V is an n-dimensional vector bundle over S and T is an endomorphism of vector bundles $V \to V$. Our equivalence of families $\sim$ can be defined such that $(V, T) \sim (V', T')$ whenever there is an isomorphism of vector bundles $\phi : V \to V'$ such that $T' \circ \phi = \phi \circ T$.

Indeed, our choice of definition for family may seem arbitrary, but in fact this is our choice of what the moduli problem actually is, enabling us to discuss what it means for elements of our set of spaces A to ‘vary geometrically’. In the end, as defined in [39, pg. 16], a moduli problem consists of a set A, an equivalence relation $\sim$ on A and a concept of a "family parameterized by some $S \in \mathbf{Var}_k$" such that:

1. A family parameterized by the single point $*$ is a single object of A.

2. The relation $\sim$ may be extended to families, which on $*$ is just the original $\sim$.

3. Given $\phi : S \to S'$ in $\mathbf{Var}_k$ and X parameterized by S', we have an induced family $\phi^*(X)$ parameterized by S. Furthermore, we should have $\phi^* \circ \phi'^* = (\phi' \circ \phi)^*$, 1_S^* should be the identity and ϕ^* should preserve $\sim$, ie. $X \sim X' \Rightarrow \phi^*(X) \sim \phi^*(X')$.

This definition can be summarized in a presheaf $\mathscr{F} : \mathbf{Var}_k^{\text{op}} \to \mathbf{Set}$ where $\mathscr{F}(S)$ is the set of equivalence classes of families $[X]$ parameterized by S and $\mathscr{F}(\phi) = \phi^*$ for $\phi : S \to S'$. We will often omit the square brackets when $[X]$ is passed as an argument, writing $\mathscr{F}(X)$ instead of $\mathscr{F}([X])$ for example.

It is in defining $\mathscr{F}$ that we perhaps spoil the ending to this story. If $\mathscr{F}$ were to be representable, having a natural isomorphism $\Phi : \mathscr{F} \to \mathrm{Hom}(-, M)$ for some $M \in \mathbf{Var}_k$, we would necessarily have that every equivalence class of families $[X]$ parameterized by some S would correspond to some unique map $\phi_{[X]} : S \to M$. In fact, M turns out to have $A/{\sim}$ as its underlying set, with $\phi_{[X]}(s) = [X_s]$ for all $s \in S$ - both telling criteria for a moduli space. These phenomena shall be justified below.

If we consider the single point variety $*$, then every equivalence class of families $[X]$ parameterized by $*$ is just an element of $A/{\sim}$, corresponding to a map $\phi_{[X]} : * \to M$. So, each point of M corresponds to a unique element of $A/{\sim}$ and we have $M \cong A/{\sim}$ as sets, via the bijection $\Phi(*)$. We will assume this is an equality henceforth for simplicity, showing that M must indeed be the moduli space we seek.

To see the second point about the maps $\phi_{[X]}$, we must assume that for any point $s : * \to S$ and any family $X \in \mathscr{F}(S)$, $X_s = s^*(X)$. We could indeed see this to be the definition of X_s in our somewhat abstract definition of a family. We will see a point s of S as an element in S or an injection $* \to S$ interchangeably.

With this in mind, Φ defines a family $[U] \in \mathscr{F}(M)$ such that $\Phi(S)(U) = 1_M$. For any family X parameterized by S, we note that

$$\begin{aligned}\phi_{[X]} = 1_M \circ \phi_{[X]} &= \mathrm{Hom}(\phi_{[X]}, M)(1_M) \\ &= \mathrm{Hom}(\phi_{[X]}, M)(\Phi(M)(U)) \\ &= \Phi(S)(\phi_{[X]}^*(U))\end{aligned}$$

the last equivalence being due to the commutative diagram

$$\begin{array}{ccc} \mathscr{F}(M) & \xrightarrow{\Phi(M)} & \mathrm{Hom}(M,M) \\ \phi^*_{[X]}\big\downarrow & & \big\downarrow \mathrm{Hom}(\phi_{[X]},M) \\ \mathscr{F}(S) & \xrightarrow{\Phi(S)} & \mathrm{Hom}(S,M) \end{array}$$

So, $\Phi(S)(X) = \Phi(S)(\phi^*_{[X]}(U))$ and we have $X \sim \phi^*_{[X]}(U)$ for any family X of any S. This property earns U the somewhat self-evident title of a *universal family* [39, pg. 19].

Regardless, if we set $S = *$, we have that $U_{\phi_{[X]}} = \phi^*_{[X]}(U) \sim X$, where we see $\phi_{[X]}$ as an element of $A/\sim$. From this, it follows that for any $X \in A$, $U_{[X]} \in [X]$ where we see $[X]$ as a point in M.

Now letting $S \in \mathbf{Var}_k$ be arbitrary, consider a point $s : * \to S$ and a family X parameterized by S. We now have, due to $X \sim \phi^*_{[X]}(U)$ and s^* preserving $\sim$,

$$\begin{aligned} X_s \sim (\phi^*_{[X]}(U))_s = s^*(\phi^*_{[X]}(U)) &= (\phi_{[X]} \circ s)^*(U) \\ &= (\phi_{[X]}(s))^*(U) \\ &= U_{\phi_{[X]}(s)} \in \phi_{[X]}(s) \end{aligned}$$

At the end, we have noted that $\phi_{[X]}(s)$ is the same as $\phi_{[X]} \circ s$ when the former is interpreted in the form $* \to M$. Thus, we finally have

$$\phi_{[X]}(s) = [X_s]$$

for all $s \in S$.

We now see that if M represents $\mathscr{F}$, then every equivalence class of families $[X] \in \mathscr{F}(S)$ parameterized by S is just a map $S \to M$, each point s being sent to the equivalence class $[X_s]$ of elements in A assigned to it by $[X]$. Indeed, we should expect a moduli space to have this property; assigning an element of A to each point on S should be, up to equivalence, the same as mapping each point on S to a point on M. This gives us our first major definition:

Definition 1.1. [39, pg. 18] A *fine moduli space* of a moduli problem $\mathscr{F}$, as defined above, is a pair (M, Φ) that represents $\mathscr{F}$.

1.1.1 Coarse Moduli Spaces

It is perhaps unfortunate that a large number of moduli problems do not have fine moduli spaces, a phenomenon we will explore soon. In the meantime however, it is possible that while there is no perfect moduli space, there may be a 'best approximation', or something *universal* in more category theoretic terms. This leads us to the notion of a *coarse* moduli space.

Definition 1.2. [39, pg. 19] A *coarse moduli space* of a moduli problem is some variety $M \in \mathbf{Var}_k$ and a natural transformation

$$\Phi : \mathscr{F} \to \mathrm{Hom}(-, M)$$

such that:

1. $\Phi(*)$ is bijective.

2. For all other varieties N and natural transformations $\Psi : \mathscr{F} \to \mathrm{Hom}(-, N)$, there is a unique natural transformation

$$\Omega : \mathrm{Hom}(-, M) \to \mathrm{Hom}(-, N)$$

   such that $\Psi = \Omega \circ \Phi$.

From the commutative diagram

$$\begin{array}{ccc} \mathscr{F}(S) & \xrightarrow{\Phi(S)} & \mathrm{Hom}(S, M) \\ {\scriptstyle [X]\mapsto[X_s]}\big\downarrow & & \big\downarrow{\scriptstyle \phi\mapsto\phi(s)} \\ A/\sim & \xrightarrow{\quad\Phi(*)\quad} & M \end{array}$$

[39, pg. 18] where $A/\sim\, = \mathscr{F}(*)$ and $M = \mathrm{Hom}(M, M)$ we have $\Phi(S)(X)(s) = \Phi(*)(X_s)$ for all $s \in S$. Hence,

$$\Phi(S)(X) = \Phi(*) \circ \phi_{[X]}$$

where $\phi_{[X]}(s) = [X_s]$ as with the case of fine moduli spaces. (The reader may note we could have used this reasoning for fine moduli spaces too.) Furthermore, if we are in the situation of part 2 in our above definition, we have a map

$$\mu = \Psi(*) \circ \Phi(*)^{-1} : M \to N$$

Note then that $\mu = \Omega(*)$. Given any variety S containing a point $s : * \to S$, we have the commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}(S,M) & \xrightarrow{\Omega(S)} & \mathrm{Hom}(S,N) \\
{\scriptstyle \mathrm{Hom}(s,M)}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Hom}(s,N)} \\
M & \xrightarrow{\quad \Omega(*)=\mu \quad} & N
\end{array}$$

This shows that $\Omega(S)(\phi)(s) = \mu(\phi \circ s)$, so $\Omega(S)(\phi) = \mu \circ \phi = \Psi(*) \circ \Phi(*)^{-1} \circ \phi$. Conversely, given a bijection $\alpha : A/{\sim} \to M$ we can define $\Phi(S)(X) := \alpha \circ \phi_{[X]}$ and get a coarse moduli space, given $\alpha \circ \phi_{[X]}$ is always a valid morphism. Hence, a coarse moduli space is also defined uniquely by a space M and the map $\Phi(*)$. We formalize this in the next definition:

Definition 1.3. [39, pg. 20] A *coarse moduli space* is a variety M and bijection $\alpha : A/{\sim} \to M$ such that:

1. For all families X parameterized by a variety S, $\alpha \circ \phi_{[X]}$ is a morphism.

2. For any variety N and natural transformation $\Phi : \mathscr{F} \to \mathrm{Hom}(-, N)$, the map

$$\mu = \Phi(*) \circ \alpha^{-1} : M \to N$$

is a morphism.

We can now prove uniqueness of coarse moduli spaces. Indeed, for any two coarse moduli spaces (M_1, α_1), (M_2, α_2), we have that $\mu = \alpha_2 \circ \alpha_1^{-1} : M_1 \to M_2$ is a morphism with inverse $\alpha_1 \circ \alpha_2^{-1}$, which we prove with Definition 1.3 and the equivalence to Definition 1.2.

Furthermore, this second definition is completely independent from the extension of $\sim$ to families. This shows that coarse moduli spaces are independent of the seemingly arbitrary choice we had to make of this extension - a reassuring statement, to be sure. This property then holds for fine moduli spaces if they exist, as any fine moduli space is coarse as well.

1.2 Existence of Moduli Spaces

With our new definitions in tow, our first rather heartbreaking order of business is to show their severe limitations. We follow the example of [39, pg. 23-24] to do this, searching for a moduli space solving the problem $\mathbf{End}_n$ we defined earlier.

Proposition 1.1. [39, pg. 24] There is no fine moduli space for $\mathbf{End}_n$.

Proof. Consider a complete variety S with a non-trivial line bundle L, such as $\mathbb{P}^1$ with the natural hyperplane bundle H. Choose an endomorphism T of the trivial bundle I_n. We claim that if there exists a fine moduli space M then (I_n, T) and $(I_n \otimes L, T \otimes 1_L)$ are non-isomorphic families such that $\phi_{[(I_n,T)]} = \phi_{[(I_n \otimes L, T \otimes 1_L)]}$.

The latter comment is not hard to see, since $(I_n, T)_s \cong (I_n \otimes L, T \otimes 1_L)_s$ for all $s \in S$ due to the map $x \mapsto x \otimes 1$. We will show the former by contradiction: if $I_n \otimes L$ were trivial for any $n \geq 1$, then L must be trivial [39, pg. 12].

If $I_n \otimes L$ were trivial, L would necessarily have a nonzero section s. Furthermore, if we consider the determinant line bundle of $I_n \otimes L$

$$\det(V) = \bigwedge_{\mathrm{rk}(V)} V$$

we see it must be the same as $\det(I_n)$ due to evident functorality. Furthermore, we must have that $\det(I_n)$ is trivial, so $\det(I_n \otimes L)$ is as well. On top of this, it is not hard to see in general, first looking at fibers then globally, that $\det(V_1 \otimes V_2) \cong \det(V_1)^{\mathrm{rk}(V_2)} \otimes \det(V_2)^{\mathrm{rk}(V_1)}$ for arbitrary vector bundles V_1, V_2 [9]. This shows that

$$\det(I_n \otimes L) \cong \det(I_n) \otimes \det(L)^n \cong I \otimes L^n \cong L^n$$

is also trivial, where $L^n = L \otimes \cdots \otimes L \cong I_1$. Since a section of I_1 is just a global section on S, it must be constant and thus if not identically zero then nonzero everywhere, meaning the same is true for every nonzero section s^n of L^n and thus for the section s of L. This gives us an isomorphism $I \to L$ defined by $(x, \lambda) \mapsto (x, \lambda s_x)$. □

The general phenomenon described here, as noted in [21, pg. 2] is that of a non-trivial *iso-trivial* family, namely a family X over a variety S where $X_{s_1} \sim X_{s_2}$ for all $s_1, s_2 \in S$ but X is not a trivial family. (In this context, a *trivial family* is a family equivalent to some X where $X_{s_1} = X_{s_2}$ for all $s_1, s_2 \in S$). We can see this in our case by setting $T = 1_{I_n}$.

To see why a non-trivial iso-trivial family X of S obliterates any hope for a fine moduli space, assume for a contradiction that M is the space we seek with universal family U. It is clear that the map $\phi_{[X]} : S \to M$ will be a constant map. Thus, the family $\phi^*_{[X]}(U)$ will be constant everywhere, making it trivial. However, $\phi^*_{[X]}(U) \sim X$, meaning X must have been trivial itself.

The reader may however find it somewhat evasive to give a proof using the arbitrary choice of families we made for this problem. Regardless, as our prior discussion about extensions of $\sim$ to families highlighted, any acceptable choice of $\sim$ for families should not have been an issue. Indeed, the problem runs even deeper than what we have described so far:

Proposition 1.2. [39, pg. 24] There is no coarse moduli space for $\mathbf{End}_n$.

Proof. Our proof completes the ideas posed in [39, pg. 24]. Consider the variety $\mathbb{A}^1_k$ and the morphism

$$\begin{aligned} f : &\mathbb{A}^1_k \to M_n(k) \\ t \mapsto B_t :&= \begin{pmatrix} 1 & 0 & \cdots & t \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \end{aligned}$$

where B_t is the matrix zero everywhere, other than 1's on its diagonal and t in the upper right corner for $t \in k$. This clearly defines a family $X \in \mathscr{F}(\mathbb{A}^1_k)$, with $\mathscr{F}$ tailored appropriately to $\mathbf{End}_n$. Note however that $B_t \sim B_1$ for all $t \neq 0$ and $B_0 \not\sim B_1$, since $\sim$ is in this case similarity. Indeed, for the former we may conjugate by the diagonal matrix whose every diagonal entry is t except the first which is 1 and for the latter $B_0 = I_n$ is similar only to itself. It follows that $\phi_{[X]}(t) = \phi_{[X]}(1)$ for all $t \neq 0$ and $\phi_{[X]}(0) \neq \phi_{[X]}(1)$.

Now, consider a coarse moduli space as in the case of Definition 1.3, namely a variety M and bijection $\alpha : A/\sim \to M$. We must have that $g = \alpha \circ \phi_{[X]} : \mathbb{A}^1_k \to M$ is a morphism. However, to be a morphism, we require continuity. If there is an open set $U \subset M$ containing $g(0)$ and not $g(1)$, we would have $g^{-1}(U) = \{0\}$ is not open, making it discontinuous. Hence, any open set containing $g(0)$ must also contain $g(1)$, meaning equivalently any closed set containing $g(1)$ contains $g(0)$. The singleton set $\{g(1)\}$ is closed due to schemes being locally Spec(R) for some R, meaning $g(0) = g(1)$. As $\phi_{[X]}(0) \neq \phi_{[X]}(1)$, this can only be because $\alpha([B_0]) = \alpha([B_1])$, contradicting α's bijectivity. □

This is another example of a more general pathological behavior called the *jump phenomenon* [39, pg. 21-22]: it is occasionally possible to construct a family X parameterized by some S where there is some $s \in S$ such that $X_q \sim X_r$ for all $q, r \neq s$ and $X_q \not\sim X_s$ for all $q \neq s$. Intuitively, there is a sudden 'jump' at the point s in the equivalence class, a phenomenon that is not geometric enough to be captured by any concrete moduli space directly.

Our example paints a somewhat disheartening picture - even something so geometrically fundamental and straightforward as the n-dimensional endomorphisms up to similarity cannot be concretely parameterized! This exact issue highlights how subtle moduli theory is in general when it comes to existence. However, the reader should keep in mind that all of the issues discussed above are due to a nontrivial equivalence relation $\sim$. Indeed, if this relation were trivial, both of the problems above would disappear and $\mathbf{End}_n$ would straightforwardly have the moduli space $M_n(k)$ itself. This will be a central focus of our discussions later.

1.3 Geometric Invariant Theory

It is commonplace in geometry that equivalences between spaces emerge from some form of *symmetry.* Indeed, if a given moduli problem $A/\sim$ has a naturally geometric relation $\sim$, such as equivalence up to rotation, translation and scaling, one should be able to easily find a group action $G \curvearrowright A$ whose orbits are precisely the equivalence classes $A/\sim$.

For instance, consider the moduli problem $A/\sim$ where $A = \mathbb{A}_k^n \setminus \{0\}$ is all the nonzero k-valued points in n-dimensional space and $x \sim y$ if and only if $x = ay$ for some $a \in k \setminus \{0\}$. This relation is clearly defined by an action $\mathbb{G}_m \curvearrowright A$ of the scalar multiplicative group on A by $t \cdot (a_1, \ldots, a_n) = (ta_1, ta_2, \ldots, ta_n)$. As from [27, pg. 18], we see the orbits are exactly the punctured lines through the origin.

If we sought out a moduli space for this problem, we would clearly wind up with the projective space $\mathbb{P}_k^{n-1}$. What relation does this have to the group action we noticed? The answer lies in the evident surjective morphism $\pi : A \to \mathbb{P}_k^{n-1}$ such that $x \sim y \Leftrightarrow \pi(x) = \pi(y)$. It is in this case that the *orbit space* $X/G := \{G \cdot x \mid x \in X\}$ [27, pg. 20] has a geometric realization which π surjects onto as a morphism in the way we expect, forming a kind of quotient. Constructing such a quotient space is clearly equivalent to finding the moduli space we sought.

Why is looking for quotients of group actions like $\mathbb{P}_k^{n-1}$ a good idea? The answer comes from Mumford's geometric invariant theory [36], which gives us the machinery needed to construct quotients or a universal approximation thereof in a wide variety of cases. The important idea Mumford contributed was a notion of 'stability', where the failure of a quotient to exist is blamed upon some orbits being too 'unstable' to include. This has to do with the size of a point's stabilizer group.

1.3.1 Orbit Spaces

We establish the formal machinery needed to take quotients of group actions, as in [27]. Our exposition will be rather skeletal, as the full theory is too much for this dissertation. Our first definition is rather general, as we will need similar generalizations later.

Definition 1.4. [31, pg. 75] Let $\mathscr{C}$ be a category with finite products and terminal object $*$, where $\alpha : x \times (y \times z) \to (x \times y) \times z$ is the canonical isomorphism and $\Delta_x : x \to x \times x$ the diagonal for all $x, y, z \in \mathscr{C}$. A *group object in* $\mathscr{C}$ is an object $g \in \mathscr{C}$ with morphisms $\mu : g \times g \to g, \nu : * \to g, \xi : g \to g$ such that the following diagrams commute:

$$
\begin{array}{ccccc}
g \times (g \times g) & \xrightarrow{\alpha} & (g \times g) \times g & \xrightarrow{\mu \times 1_g} & g \times g \\
{\scriptstyle 1_g \times \mu}\downarrow & & & & \downarrow{\scriptstyle \mu} \\
g \times g & & \xrightarrow{\qquad\quad \mu \qquad\quad} & & g
\end{array}
$$

$$
\begin{array}{ccccc}
* \times g & \xrightarrow{\nu \times 1_g} & g \times g & \xleftarrow{1_g \times \nu} & g \times * \\
 & {\scriptstyle \mathrm{pr}_2}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle \mathrm{pr}_1} & \\
 & & g & &
\end{array}
$$

$$
\begin{array}{ccccccccc}
g & \xrightarrow{\Delta_g} & g \times g & \xrightarrow{1_g \times \xi} & g \times g & \xleftarrow{\xi \times 1_g} & g \times g & \xleftarrow{\Delta_g} & g \\
\downarrow & & & & \downarrow{\scriptstyle \mu} & & & & \downarrow \\
* & & \xrightarrow{\qquad \nu \qquad} & & g & & \xleftarrow{\qquad \nu \qquad} & & *
\end{array}
$$

Note in [31] the above is called simply a 'group in $\mathscr{C}$'. The reader should clearly consider μ to be group multiplication, ν to represent the 'identity element' of g and ξ to be inversion. Note that Mac Lane's definition only requires right inverses, whereas we make two-sided inverses explicit as in [27, pg. 12] for clarity. We will spare the reader the laborious but straightforward diagrammatic proof that these are equivalent.

We give a definition for group homomorphism generalizing the one in [27, pg. 12]:

Definition 1.5. A *group homomorphism* $f : g \to h$ is a morphism between groups $g, h \in \mathscr{C}$ such that the diagram

$$
\begin{array}{ccc}
g \times g & \xrightarrow{f \times f} & h \times h \\
{\scriptstyle \mu_g}\downarrow & & \downarrow{\scriptstyle \mu_h} \\
g & \xrightarrow{f} & h
\end{array}
$$

commutes, where μ_g, μ_h are the appropriate multiplication morphisms for g and h respectively.

Definition 1.6. [27, pg. 12] An *algebraic group over* k for some field k is a group object in the category of k-varieties $\mathbf{Var}_k$.

The above definition in [27] defines algebraic groups to be group objects in the more general $\mathbf{Sch}_k$, but this distinction does not matter too much to us; we are only giving an overview of this topic. Algebraic group homomorphisms will be group homomorphisms in $\mathbf{Var}_k$ as well.

[27] gives a definition of algebraic group action on a scheme, which we choose to generalize first.

Definition 1.7. Let g be a group object in $\mathscr{C}$ and $x \in \mathscr{C}$. A *group action of g on x* is a morphism $\sigma : g \times x \to x$ such that the following diagrams commute:

$$\begin{array}{ccc} * \times x & \xrightarrow{\nu \times 1_x} & g \times x \\ & \searrow^{\cong} & \downarrow{\scriptstyle \sigma} \\ & & x \end{array} \qquad \begin{array}{ccccc} (g \times g) \times x & \xleftarrow{\alpha} & g \times (g \times x) & \xrightarrow{1_g \times \sigma} & g \times x \\ \downarrow{\scriptstyle \mu \times 1_x} & & & & \downarrow{\scriptstyle \sigma} \\ g \times x & & \xrightarrow{\quad\quad \sigma \quad\quad} & & x \end{array}$$

Definition 1.8. Let $\sigma_x : g \times x \to x$ and $\sigma_y : g \times y \to y$ be group actions of g on x and y in $\mathscr{C}$, respectively. A morphism $f : x \to y$ is called *g-equivariant* if the following diagram commutes:

$$\begin{array}{ccc} g \times x & \xrightarrow{1_g \times f} & g \times y \\ \downarrow{\scriptstyle \sigma_x} & & \downarrow{\scriptstyle \sigma_y} \\ x & \xrightarrow{f} & y \end{array}$$

Definition 1.9. Let G be an algebraic group over k and X a k-variety. Then a group action of G on X is a group action in $\mathbf{Var}_k$. A G-equivariant map is defined similarly.

We could have easily skipped this formality and just declared an algebraic group to be a variety with an appropriate group structure. Our approach here is really a mental exercise to prepare for later, such as when we discuss groupoid objects. In the specific case of algebraic groups, orbits $G \cdot x$ and stabilizers G_x are defined pointwise as usual for all $x \in X$.

Regardless, we are now able to discuss quotients of group actions. We start somewhat in reverse to our approach with moduli spaces, beginning not with an idealized quotient but with a universal approximation thereof.

Definition 1.10. [27, pg. 20] Let G be an algebraic group over k acting on a scheme X. Then a *categorical quotient* of this action is a G-equivariant morphism $\phi : X \to Y$ such that every other G-equivariant $f : X \to Z$ has a unique G-equivarant $h : Y \to Z$ such that $f = h \circ \phi$. Furthermore, if $\phi^{-1}(\{x\})$ is a single orbit for each $x \in Y$, we call Y an *orbit space.*

From now on, we take 'orbit space' to mean this definition rather than the set we wrote earlier, as in [39] and [27]. The reader should also keep in mind that we may say something is a categorical quotient or orbit space without mentioning the morphism ϕ, as it is usually obvious.

1.3.2 Connections to Moduli Theory

Why would we want to define a categorical quotient? As we have seen and are about to formalize, this definition is in many ways comparable to that of a coarse moduli space. We will follow [39, pg. 30-31] to this end.

Definition 1.11. [39, pg. 30] Let $A/\sim$ be a moduli problem. A family X parameterized by S in this problem has the *local universal property* exactly when, for any family X' parameterized by S' and $s \in S'$, there exists a neighborhood U of s such that $X'|_U \sim f^*(X)$ for some $f : U \to S$, where $X'|_U$ is the pullback along the inclusion $U \hookrightarrow S$. We say S has the local universal property similarly if it has such a family X.

An example of such a family can be found in the case of $\mathbf{End}_n$, where we take the family F induced from $1_{M_n(k)}$ parameterized by $M_n(k)$. To prove this has the property we seek, let (V, T) be a family parameterized by S. Should we have $V \cong I_n$ then the family corresponds to a morphism $f : S \to M_n(k)$, which we may pull back F with to see $(V, T) \sim f^*(F)$. Since every vector bundle is locally trivial, we are finished.

It should be noted that in the above definition, S will surject onto $A/\sim$; indeed, every element A is equivalent to the pullback of X along a map $* \to S$. Of course, this will usually not be a bijection.

We now come to our first theorem:

Theorem 1.1. [39, pg. 32] Let $A/\sim$ be a moduli problem. Suppose there is some family X parameterized by S with the local universal property. Furthermore, suppose we have an algebraic group G acting on S such that $X_s \sim X_t$ if and only if $G \cdot s = G \cdot t$. Then:

1. Any coarse moduli space is a categorical quotient of S by G.

2. A categorical quotient of S by G is a coarse moduli space precisely when it is an orbit space.

Proof. We will show there to be a bijection from morphisms $\phi : S \to M$ constant on orbits to natural transformations $\Phi : \mathscr{F} \to \mathrm{Hom}(-, M)$, where $\mathscr{F}$ is the presheaf of families for our moduli problem.

Define $\phi = \Phi(S)(X)$. Our assumptions about G's actions on X imply this map will be constant on orbits. Conversely, given $\phi : S \to M$ constant on orbits and a family X' parameterized by S', it is clear that X' can be covered by open sets U_i such that $X'|_{U_i} \sim \psi_i^*(X)$ for some $\psi_i : U_i \to S$. We can construct a corresponding natural transformation $\Phi : \mathscr{F} \to \mathrm{Hom}(-, M)$ by

$$\Phi(S')(X')|_{U_i} = \phi \circ \psi_i$$

for all U_i. To see why this is well-defined, we must show compatibility on overlaps, ie. that $(\phi \circ \psi_i)|_{U_i \cap U_j} = (\phi \circ \psi_j)|_{U_i \cap U_j}$ for all i, j. We begin with the observation that $\psi_i^*(X)|_{U_i \cap U_j} \sim \psi_j^*(X)|_{U_i \cap U_j} \sim X'|_{U_i \cap U_j}$, which implies $X_{\psi_i(s)} \sim X_{\psi_j(s)}$ for all $s \in U_i \cap U_j$. By our assumptions we deduce that $\psi_i(s)$ and $\psi_j(s)$ are in the same orbit, leading us to conclude $\Phi(S')(X')$ is well-defined as needed. This is of course independent of the choice of covering U_i, as another viable covering V_j lets us produce an intersection covering $U_i \cap V_j$ that leads to a Φ coinciding locally with both the result of using U_i and the result of using V_j.

Furthermore, we can show Φ is natural by proving that for any $f : Q \to R$ the diagram

$$\begin{array}{ccc} \mathscr{F}(R) & \xrightarrow{\Phi(R)} & \mathrm{Hom}(R, M) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}(f,M)} \\ \mathscr{F}(Q) & \xrightarrow{\Phi(Q)} & \mathrm{Hom}(Q, M) \end{array}$$

commutes. For an arbitrary family F parameterized by R, this amounts to showing that $\Phi(Q)(f^*(F)) = \Phi(R)(F) \circ f$. Let V_i be the covering of R with maps ψ_i and U_i be the covering of Q with δ_i we choose in our above definitions. Then for any $s \in U_i \subset Q$ with a choice of V_j such that $f(s) \in V_j \subset R$, we have $\Phi(Q)(f^*(F))(s) = (\phi \circ \delta_i)(s)$ and $(\Phi(R)(F) \circ f)(s) = (\phi \circ \psi_j)(f(s))$. Hence, the maps are equal exactly if for every $s \in Q$, there are open sets in our covers $s \in U_i$ and $f(s) \in V_j$ such that $\psi_j(f(s))$ and $\delta_i(s)$ are in the same orbit.

Proving this is equivalent to showing $X_{\psi_j(f(s))} \sim X_{\delta_i(s)}$, which holds if and only if $(\psi_j \circ f)^*(X)_s \sim \delta_i^*(X)_s$. We can now prove Φ is indeed a natural transformation by recalling $((\psi_j \circ f)^*(X))_s = f^*(\psi_j^*(X))_s \sim f^*(F)_s \sim \delta_i^*(X)_s$.

We now show that $\phi \mapsto \Phi$ and $\Phi \mapsto \phi$ are mutually inverse. The direction $\phi_1 \mapsto \Phi \mapsto \phi_2$ is not hard, since for the family X of S we can have a covering U_i of one set, namely S, with $\psi_1 = 1_S$ so that $\Phi(S)(X)|_S = \phi_1 \circ 1_S = \phi_2$. For the $\Phi_1 \mapsto \phi \mapsto \Phi_2$ direction, given a covering U_i and maps ψ_i for a family X' parameterized by S', we clearly have

$$\begin{aligned}\Phi_2(S')(X')|_{U_i} = \phi \circ \psi_i &= \Phi_1(S)(X) \circ \psi_i \\ &= \Phi_1(U_i)(\psi_i^*(X)) \\ &= \Phi_1(U_i)(X'|_{U_i}) \\ &= \Phi_1(S)(X')|_{U_i}\end{aligned}$$

where the last few equalities are due to Φ_1 being a natural transformation. We thus have our bijection.

Completing the proof is now straightforward, as we simply observe that (M, ϕ) is now a categorical quotient exactly when (M, Φ) satisfies property 2 in Definition 1.2, our bijection giving the unique factorizations on either end. It is then clear that (M, Φ) satisfies property 1 of Definition 1.2 if and only if (M, ϕ) is an orbit space. □

As an example, $\mathbf{End}_n$ satisfies the criteria for this theorem if we let $S = M_n(k)$, $X = F$ and $G = \mathrm{GL}_n(k)$ act by conjugation [39, pg. 33].

This theorem gives us strong evidence that solving moduli problems by locating a family with the local universal property and searching for either a categorical quotient or orbit space is a viable approach to solving many moduli problems, if our relation $\sim$ is defined by an algebraic group action as it often is.

1.3.3 Stability

We have defined the GIT approach to moduli theory and explained its relevance. Now, we demonstrate its practicality through Mumford's notion of stability. The goal of this section is to, given an algebraic group action of G on X, discover open subsets $X^s \subset X^{ss} \subset X$ of special 'stable' and 'semistable' points that always admit an orbit space and categorical quotient, respectively. This will leave behind a closed set of 'unstable' points, all having some unnatural properties that justify their exclusion.

We may be frustrated by a solution that effectively dodges the heart of the problem, but if we are indeed fixated on finding a concrete quotient or moduli space, it is these somewhat unsatisfying sacrifices that we are forced to make. It is on occasion possible to analyze the unstable points through other means, such as using Harder-Narasimhan filtrations to subdivide the set of unstable vector bundles over a smooth projective curve into more digestible fragments [24], but we can never solve the whole puzzle at once, always breaking one end when we fix another. We will have to grin and bear it for now.

To begin, we will have to introduce the notion of a reductive group, whose technical substance is too vast to explain in full detail here.

Definition 1.12. [36, pg. 24] Let G be an algebraic group. A *representation* of G is an algebraic group homomorphism $G \to \mathrm{GL}_n(k)$ for some n. The induced action of G on $\mathbb{A}^n_k$ is called a *linear action* [39, pg. 37].

If there exists a representation of G that is also a closed immersion, then we call G a *linear algebraic group* [39, pg. 37].

Examples of linear algebraic groups include $\mathrm{SL}_n(k)$, $\mathrm{PGL}_n(k)$ and the multiplicative group $\mathbb{G}_m = k^* = \mathrm{GL}_1(k)$.

Mumford's definition of a representation in [36, pg. 24] notes that there are in fact three distinct actions we could have chosen to induce: our action on the affine variety $\mathbb{A}^n_k$, a linear action on k^n as a vector space and a dual action on the global sections. While we won't go into the details of this last action (see [39, pg. 39] and [36, pg. 25]), it essentially amounts to an action on the functions on $\mathbb{A}^n_k$ and in general the functions $X \to k$ over any X that G acts on. We see geometric substance in this statement since, for any categorical quotient (Y, ϕ) of X by this action, we have that only those morphisms $X \to k$ constant on orbits may factor through ϕ. Hence, the induced function on global sections

$$\phi^* : \mathscr{O}_Y(Y) \to \mathscr{O}_X(X)$$

must map isomorphically onto the G-invariant functions $\mathscr{O}_X(X)^G \subset \mathscr{O}_X(X)$ [39, pg. 39]. We can therefore deduce that, for affine X, a categorical quotient Y could only hope to be affine in turn if $\mathscr{O}_X(X)^G$ was finitely generated.

The more abstract ring-theoretic form of this issue was significant enough to become Hilbert's Fourteenth Problem. The answer is, as is often the case, not so simple: Nagata proved its falsehood in general [38], but showed it to be true if G was 'reductive' [37], a property we define below.

Definition 1.13. [39, pg. 26] An algebraic group G is called *geometrically reductive* if it is linear and, for every linear action on $\mathbb{A}^n_k$ and every nonzero invariant point $v \in \mathbb{A}^n_k$, there is an invariant homogeneous polynomial f of nonzero degree such that $f(v) \neq 0$. If we can force f's degree to always be 1 then G is *linearly reductive.*

It should also be noted that Newstead in [39] reserves the term 'reductivity' for when G's radical is a torus group, but admits later that these were proven to be equivalent in [23], as conjectured by Mumford. Hence, we may use this definition for reductivity without issue. Also, Mumford's definition for reductivity in [36, pg. 24] is in fact equivalent to Newstead's definition of linear reductivity in [39], where he instead demands *complete reducibility*, ie. if the linear action of G on $\mathbb{A}_k^n$ leaves some linear subspace $\mathbb{A}_k^r \subset \mathbb{A}_k^n$ through 0 invariant then it leaves a complementary $\mathbb{A}_k^{n-r} \subset \mathbb{A}_k^n$ invariant as well. This is in general only equivalent to geometric reductivity in characteristic 0, but simply using geometric reductivity instead is claimed by Newstead to leave the arguments in [36] largely unchanged [39, pg. 41-42].

A number of important groups like $\mathrm{GL}_n(k)$, $\mathrm{SL}_n(k)$ and $\mathrm{SO}_n(k)$ are indeed reductive in this context, by the comments in [36, pg. 27] on Nagata's classification of reductive algebraic groups.

While we won't have time to fully explore the significance of this definition, for which we refer the reader to [36, ch. 1], [39, ch. 3] and [27], we will regardless present a major theorem that helps highlight the importance of reductive groups.

Definition 1.14. [27, pg. 30] Given an affine reductive algebraic group G acting on an affine X, the *affine GIT quotient* is the morphism $\phi : X \to X//G := \mathrm{Spec}(\mathscr{O}_X(X)^G)$, whose induced map on sheaves is the inclusion $\phi^* : \mathscr{O}_X(X)^G \hookrightarrow \mathscr{O}_X(X)$.

It is clear from our discussions that the affine GIT quotient is indeed affine.

Theorem 1.2. [39, pg. 51] In the context of Definition 1.14, the affine GIT quotient $\phi : X \to X//G$ satisfies the following:

1. ϕ is G-invariant.
2. ϕ is surjective.
3. If U is open in $X//G$ then $\phi^* : \mathscr{O}_{X//G}(U) \to \mathscr{O}_X(\phi^{-1}(U))$ is an isomorphism of $\mathscr{O}_{X//G}(U)$ onto $\mathscr{O}_X(\phi^{-1}(U))^G$.
4. If W is a closed invariant subset of X then $\phi(W)$ is closed.
5. If W_1, W_2 are disjoint closed invariant subsets of X then $\phi(W_1) \cap \phi(W_2) = \emptyset$.

In this context, an invariant subset is one that contains all its elements' orbits. A proof of this is given in [39, pg. 51] and in [27, pg. 31].

In general, any affine morphism $X \to Y$ given an action of an algebraic group G on X that satisfies properties 1-5 above is called a *good quotient* [27, pg. 20-21]. It is proven in [27, pg. 21] that any good quotient is a categorical quotient. If we further require that Y is an orbit space then we have a *geometric quotient* [27, pg. 21].

We list a further proposition without proof from [39, ch. 3], simply to paint an interesting picture of perhaps why good and geometric quotients are interesting.

Proposition 1.3. [39, pg. 55] Let (Y, ϕ) be a good quotient of the action of G on X and define

$$X' = \{x \in X \mid G \cdot x \text{ is closed and } \dim(G \cdot x) \text{ is maximal}\}$$

Then there is an open subset $Y' \subset Y$ such that $X' = \phi^{-1}(Y')$ and (Y', ϕ) is a geometric quotient of X' by G.

It is not hard to visualize why having a categorical quotient where the orbits are of variable dimension would be difficult to arrange into an orbit space. For instance, an invariant point surrounded completely by a higher-dimensional orbit would be identified with said orbit in the quotient for similar reasons to the jump phenomenon. Further, if orbits are not closed, two orbits which do not intersect but whose closures do will be identified in the quotient, as the preimage of a single point must be closed.

We will now turn our attention to the case where X is a projective variety in $\mathbb{P}^n_k$ acted on by a reductive G. Given a *linearisation* of the action of G on X, namely an action on $\mathbb{A}^{n+1}_k$ that descends to the action on X via its affine cone, we obtain the following definitions:

Definition 1.15. [39, pg. 60] A point $x \in X$ is called

1. *semistable* if there is an invariant homogeneous polynomial f of degree ≥ 1 such that $f(x) \neq 0$;
2. *stable* if $\dim(G \cdot x) = \dim(G)$ and there is an invariant homogeneous polynomial f of degree 1 such that $f(x) \neq 0$ and the action of G on X_f is closed.

The stable and semistable points reside in sets $X^s \subset X^{ss} \subset X$, respectively. As noted in [39, pg. 61], our notion of stability is actually *proper stability* in [36, pg. 37], where stability does not demand maximal dimension of the orbit. [36] also defines a notion of *pre-stability* where we only demand an invariant affine open subset $x \in U \subset X$ where the action on U is closed. Further, Mumford's notion of stability does not depend on the embedding of X in $\mathbb{P}^n$ as we have implicitly done, preferring to use ample line bundles to do the job. We will disregard these somewhat distracting discrepancies here, as we proceed to what Newstead refers to in his book as "...the central result of these notes..." [39, pg. 61]:

Theorem 1.3. [39, pg. 61] Let X be a projective variety in $\mathbb{P}^n$. Then, given an action of a reductive group G on X with a linearisation:

1. There exists a good quotient (Y, ϕ) of X^{ss} such that Y is projective.
2. There exists an open subset Y^s of Y such that $\phi^{-1}(Y^s) = X^s$ and (Y^s, ϕ) is a geometric quotient of X^s.
3. For $x_1, x_2 \in X^{ss}$, $\phi(x_1) = \phi(x_2) \Leftrightarrow \overline{G \cdot x_1} \cap \overline{G \cdot x_2} \cap X^{ss} \neq \emptyset$.
4. For $x \in X^{ss}$, x is stable if and only if $\dim(G \cdot x) = \dim(G)$ and $G \cdot x$ is closed in X^{ss}.

We omit the proof here; this is given in [39, pg. 61]. This particular formulation of the major theorem is not done in [27], [36] or [34], where instead the *projective GIT quotient* of X^{ss} is defined more concretely as

$$X^{ss}(L) \to X//_L G = \mathrm{Proj} \bigoplus_{r \geq 0} H^0(X, L^{\otimes r})^G$$

where L is an ample line bundle giving the linearisation of X. To see how this works and how one defines $X^{ss}(L)$ as a space dependent on L rather than the X^{ss} we have defined, the reader is referred to [27, pg. 42] and [34, ch. 6].

We have now seen methods to construct what are, in some sense, the best possible categorical quotients and orbit spaces we can get in the affine and projective cases. This leads to an effective general pattern of constructing moduli spaces [27, pg. 11]:

1. Fix discrete invariants of the equivalence relation, such as n in $\mathbf{End}_n$.
2. Restrict to a reasonable bounded class of objects, ie. one where there exists a family F over a scheme S which parameterizes all objects in the moduli problem.
3. Choose a family X parameterized by some S with the local universal property.
4. Find an algebraic group action $G \curvearrowright S$ where $X_s \sim X_t$ if and only if $G \cdot s = G \cdot t$.
5. If the objects we have restricted ourselves to are stable under the action of G, the corresponding geometric quotient will yield a moduli space.

While this does indeed generate moduli spaces, the reader should remain wary that these spaces can only answer our original major question for confined subsets of whatever collection of spaces $A/\sim$ catches our eye. Finding geometric information about all elements of A at once will require a radical change in perspective.

Chapter 2

From Spaces to Stacks

"It is not the answer that enlightens, but the question."

– Eugène Ionesco

Our attempts to fully characterize the geometric substance of a moduli problem have been, so far, incomplete. The moduli space approach seems fundamentally incapable of giving geometric information about the entirety of a general moduli problem, a fact owing to a veritable zoo of pathological behaviors we are forced to contend with. However, if the reader has been paying close attention, they will realize that every insurmountable obstacle we have thus far run into has been completely due to the behavior of the equivalence relation $\sim$. Indeed, Grothendieck was fully aware of this conundrum, as evidenced by his letter to Serre on November 5, 1959 (translated in [6, pg. 94]):

> I have already come to the practical conclusion that every time my criteria show that no moduli variety (or rather, moduli scheme) for the classification of (global or infinitesimal) variations of certain structures (complete non-singular varieties, vector bundles etc.) can exist, despite good hypotheses of flatness, properness, and if necessary non-singularity, the only reason is the existence of automorphisms of the structure which prevent the descent from working.

It was realized at this point that a new approach would be needed to analyzing moduli problems, one that would deal with the subtle complexities of automorphisms somewhat more delicately than simply taking equivalence classes. Perhaps we now realize that our definition of the presheaf $\mathscr{F}$ is disasterously heavy-handed in this respect, suggesting that our investigation was doomed from the start. We will now attempt to rectify this poor formalization of moduli problems, including in our 'question' $\mathscr{F}$ information about the interconnected web of family equivalences rather than simply taking path components.

What may surprise the reader is that this will immediately yield the mathematical object we had desired! In posing our question $\mathscr{F}$ correctly, we shall notice that it can be investigated directly for many of the geometric properties of our moduli problem we seek. Mumford had considered such objects in 1963 [35] before they had been formalized by Giraud in 1966 [18]. In keeping with the French school's fondness for agricultural terminology Giraud referred to them as 'champs', meaning 'fields', which Deligne and Mumford later changed to *stacks* in 1969 [8]. The objects Deligne and Mumford studied are now called *Deligne-Mumford stacks*, which have since been generalized by Artin in 1975 to *algebraic stacks*, or *Artin stacks* [3].

It turns out that Artin stacks are the correct next step towards answering our original question in the context of algebraic geometry. Our goal in this chapter is to present the foundations of classical stack theory from the ground up, starting with Grothendieck topologies and finally reaching a variety of different kinds of stack. We will also see that problems we faced in GIT can be avoided by taking so-called *quotient stacks*, which will be related to Artin stacks similarly to the connection between orbit spaces and moduli spaces.

2.1 Grothendieck Topologies

Before all else, we must address the problem of *descent*. The reader may recall the notion of a structure sheaf from scheme theory, where one creates a presheaf of rings over the category of open subsets of a space X and imposes a *local-to-global* condition. This last condition explains how data on overlapping regions can be glued together into one datum. Indeed, given a space X with a covering U_i and the morphism

$$\amalg_i U_i \to X$$

that is the coproduct of the inclusions $U_i \hookrightarrow X$, a sheaf lets us understand how data in $\amalg_i U_i$ *descends* to data in X.

It is however somewhat restrictive to limit ourselves to such a small and focused category; we would like to define sheaves in a setting like $\mathbf{Sch}_k$, so that sheaves of different spaces can be compared and operated upon. Furthermore, we may be interested in sheaves defined across all spaces, such as the sheaf represented by a classifying space. Achieving this will require us to redefine 'topology' in a more palpable form for general category theory.

Our first attempt at this may be to consider subsets of monomorphisms $U_i \to X$ that represent coverings. To encode overlaps, we may demand that these subsets are closed under pullbacks, which straightforwardly corresponds to intersection in **Set** and **Top**. However, as noted in [32, pg. 108], this approach is not particularly functorial: Grothendieck encountered issues with regards to a duality between covering spaces and field extensions, where covering spaces were 'split' over monomorphisms but field extensions were not dually split over epimorphisms. This led Grothendieck to consider a notion of topology that did not force maps $U_i \to X$ in a covering to be monic, opening up a wide range of new topologies that are far more natural to category theory.

We assume from here on that our category $\mathscr{C}$ is locally small and equipped with all pullbacks, following the presentation of [32, ch. 3]. Take some $X \in \mathscr{C}$ and consider a set $S = \{f_i : U_i \to X \mid i \in I\}$ for some $U_i \in \mathscr{C}$. We could make a choice of such sets $K(X) = \{S, S', S'', \ldots\}$ for each X, for which we could foreseeably repeat the usual definition of a sheaf, starting with a presheaf $P : \mathscr{C}^{op} \to \mathbf{Set}$.

Recall in the case of sheaves over a scheme X, we demanded that for an open cover U_i of some open $U \subset X$, a choice of sections $x_i \in P(U_i)$ which matched on overlaps $P(U_i \cap U_j)$ could be seen as restrictions of a unique $x \in P(U)$. Now, we can generalize intersection to pullback, as shown in the diagram

$$\begin{array}{ccc} U_i \times_X U_j & \xrightarrow{h_{ij}} & U_j \\ {\scriptstyle v_{ij}}\downarrow & & \downarrow{\scriptstyle f_j} \\ U_i & \xrightarrow{f_i} & X \end{array}$$

which then translates into a diagram

$$\begin{array}{ccc} P(U_i \times_X U_j) & \xleftarrow{h_{ij}^*} & P(U_j) \\ {\scriptstyle v_{ij}^*}\uparrow & & \uparrow{\scriptstyle f_j^*} \\ P(U_i) & \xleftarrow{f_i^*} & P(X) \end{array}$$

Our matching condition can now be phrased as follows: given some choice of sections $x_i \in P(U_i)$ such that $v_{ij}^*(x_i) = h_{ij}^*(x_j)$ for all i and j, there exists a unique $x \in X$ such that $x_i = f_i^*(x)$ for all $i \in I$. This could alternatively be phrased as requiring e to be an equalizer in the diagram

$$P(X) \xrightarrow{e} \prod_i P(U_i) \rightrightarrows \prod_{i,j} P(U_i \times_X U_j)$$

where $e(x) = (f_i^*(x) : i \in I)$ and the rightmost maps are $(x_i : i \in I) \mapsto (v_{ij}^*(x_i) : i, j \in I \times I)$ and $(x_i : i \in I) \mapsto (h_{ij}^*(x_j) : i, j \in I \times I)$ [32, pg. 109].

Now, in order to develop our generalized notion of topology, we must simply decide what $K(X)$ should be for each $X \in \mathscr{C}$. To do this, we should consider what we really mean by a 'covering', in our transition from a topology defined by open sets to one by coverings. One may interpret a morphism $Y \to X$ as a *generalized point* in X: if $Y = *$ it is a regular point in X, if $Y = S^1$ it is a loop in X, if $Y = \mathbb{R}$ it is a path in X and so on. With this in mind, a covering should be some collection of generalized points of X that 'cover' it in some sense varying with application.

From this intuition, for any 'covering' $S \in K(X)$ and any $f_i : U_i \to X \in S$, if there is a morphism $g : V \to U_i$, then we will choose to allow $f_i \circ g \in S$. The intuition here is that S acts as a 'filter', only allowing generalized points that represent the covering it depicts. This motivates the concept of a *sieve* we will use; as stated by [32, pg. 111], "V goes through the sieve if it fits through one of the holes U_i of the sieve".

It is now possible to see S as a presheaf where $S(U) = \mathrm{Hom}(U, X) \cap S$ for all U. This could alternatively be stated by declaring S to be a subobject of the Yoneda presheaf $\mathrm{Hom}(-, X)$, which will be our formal definition of a sieve [32, pg. 37]. If we choose to see sieves as right ideals under composition, we could easily generate a sieve from any $h : Y \to X$ and sieve $S \in K(X)$ of the form

$$\widehat{h}(S) = \{g \mid \mathrm{cod}(g) = Y, h \circ g \in S\}$$

(Our notation differs from [32], since theirs collides with our notation for pullbacks of morphisms along a presheaf P.) We will demand that $\widehat{h}(S) \in K(Y)$, since all the ways the generalized points in S factor through h should cover Y.

Furthermore, consider an arbitrary sieve R on X and some $S \in K(X)$, such that for all $f_i : U_i \to X$ in S we have $\widehat{f_i}(R) \in K(U_i)$. Intuitively this represents a sieve that covers every generalized point in the covering S, so we should consequently regard it as a covering on X itself and declare $R \in K(X)$. In fact, the properties that we have discussed so far are precisely what will define our new notion of topology.

Definition 2.1. [32, pg. 110] A (Grothendieck) topology on a category $\mathscr{C}$ is a mapping J that assigns to each $X \in \mathscr{C}$ a collection $J(X)$ of sieves on X, such that

1. The maximal sieve $t_X = \{f \mid \mathrm{cod}(f) = X\}$ is in $J(X)$.
2. (Stability) If $S \in J(X)$, then $\widehat{h}(S) \in J(Y)$ for any $h : Y \to X$.
3. (Transitivity) If $S \in J(X)$ and R is any sieve on X such that $\widehat{h}(R) \in J(Y)$ for all $h : Y \to X$ in S, then $R \in J(X)$.

Definition 2.2. [32, pg. 110] A *site* is a category $\mathscr{C}$ with a topology J.

It is noted in [32] that this implies if $S \in J(X)$ is a subobject of some other sieve R, then $R \in J(X)$. Indeed, let $h : Y \to X \in S$. Then we have $1_Y \in \widehat{h}(S)$, so $\widehat{h}(S)$ is the maximal sieve on Y, meaning $\widehat{h}(R) \supseteq \widehat{h}(S)$ is equal, so must be in $J(Y)$. As this holds for all $h \in S$, transitivity implies $R \in J(X)$.

As an aside, our choice of terminology for 'mapping' J is perhaps somewhat suspect. Mac Lane and Moerdijk say 'function', but this implies $\mathscr{C}$ is small. We could replace this with a presheaf such that $J(h) = \widehat{h}$, but this would require $J(X)$ to be a set for each X, something we cannot directly guarantee even though we have assumed $\mathscr{C}$ to be locally small. Thankfully, even though approach of [53, Tag 00VI] assumes $\mathscr{C}$ is in fact small, it still addresses that in the large case we can replace each proper class of coverings with a set that yields an equivalent category of sheaves [53, Tag 000X] [53, Tag 00VY]. We will therefore carry the assumption that $J(X)$ is small throughout.

Interestingly, even in the extreme case where $\mathscr{C}$ does not have pullbacks, defining a sheaf is still feasible, as presented in [32, pg. 121-122]. Let $P : \mathscr{C}^{op} \to \mathbf{Set}$ be a presheaf over the site $\mathscr{C}$. For some $S \in J(X)$, a *matching family* for S of elements of P is choice of an element $x_f \in P(Y)$ for each $f : Y \to X$ in S such that $g^*(x_f) = x_{f \circ g}$ for all $g : Z \to Y$. An *amalgamation* of a matching family is a single element $x \in P(X)$ such that $f^*(x) = x_f$ for all $f \in S$. Then P is a sheaf exactly when every matching family of every $S \in J(X)$ has a unique amalgamation.

Note that there are alternative ways to phrase this definition. [32] points out that P is a sheaf if and only if, for any $S \in J(X)$ seen as a subobject of $\mathrm{Hom}(-, X)$, the natural transformation $S \to P$ defined by $f \mapsto x_f$ for some matching family factors uniquely through the Yoneda presheaf $\mathrm{Hom}(-, X)$. It is not hard to see that this essentially amounts to picking a unique amalgamation, ie. some x_{1_X}.

We can also express the sheaf condition diagrammatically [32, pg. 122] by stating e is an equalizer in the diagram

$$P(X) \xrightarrow{e} \prod_{f \in S} P(\mathrm{dom}(f)) \underset{a}{\overset{p}{\rightrightarrows}} \prod_{\substack{f,g,\ f \in S \\ \mathrm{dom}(f) = \mathrm{cod}(g)}} P(\mathrm{dom}(g))$$

where we define $e(x) = (f^*(x))_{f \in S}$ and the rightmost functions as $p((x_f)_{f \in S}) = (x_{f \circ g})_{f,g}$ and $a((x_f)_{f \in S}) = (g^*(x_f))_{f,g}$.

These constructions let us define a category of sheaves $\mathbf{Sh}(\mathscr{C})$ over any site $\mathscr{C}$. Interestingly, the inclusion functor into the category of presheaves $\mathbf{Sh}(\mathscr{C}) \to \mathbf{Ps}(\mathscr{C})$

always has a left adjoint, which we call the *sheafification* operation [53, Tag 00W1]. The proof is lengthy and not enlightening, so we won't replicate it here, though it ensures that any presheaf has a universal sheaf associated to it. In essence, the collection $J(U)$ of coverings of U can be made into a category by adding an arrow from any covering to any other contained within it. For any presheaf P, a functor can be made from this category to **Set**, sending a covering to all its matching families. One then defines the sheaf P^+ such that $P^+(U)$ is the colimit of this functor. The sheafification is then $(P^+)^+$, since $(-)^+$ maps presheaves to *separated presheaves*, those presheaves where amalgamations are unique when they exist, and separated presheaves to sheaves [53, Tag 00WB].

Before we close off our discussion of sites, we will briefly mention that a general covering of a topological space X is usually not defined as a sieve, but as a set of inclusions $U_i \hookrightarrow X$ that together surject onto X. Such a covering can clearly *generate* a sieve however, a phenomenon that we will find useful in defining a basis for a Grothendieck topology:

Definition 2.3. [32, pg. 111] A *basis* for a Grothendieck topology on a category $\mathscr{C}$ with pullbacks is a mapping K of objects $X \in \mathscr{C}$ to collections $K(X)$ of families of morphisms with codomain X such that

1. If $f : X' \to X$ is an isomorphism then $\{f\} \in K(X)$.
2. (Stability) If $\{f_i : U_i \to X \mid i \in I\} \in K(X)$, then for any morphism $g : Y \to X$, we have the collection of pullbacks $\{f'_i : U_i \times_X Y \to Y \mid i \in I\} \in K(Y)$.
3. (Transitivity) If $\{f_i : U_i \to X \mid i \in I\} \in K(X)$ and if for each $i \in I$ we have a collection $\{g_{ij} : V_{ij} \to U_i \mid j \in J_i\} \in K(U_i)$, then we have the collection of composites $\{f_i \circ g_{ij} : V_{ij} \to X \mid i \in I, j \in J_i\} \in K(X)$.

A basis K then generates a Grothendieck topology J where for all sieves S over X, $S \in J(X) \Leftrightarrow \exists R \in K(X), R \subseteq S$ [32, pg. 112].

2.1.1 Sites in Algebraic Geometry

After all this work to define an abstract notion of topology, which topologies are we actually interested in? It is clear perhaps that we could now choose our topologies to be things generated by the coverings we already know, such as the topology generated by coverings of Zariski open subsets $U_i \hookrightarrow X$ on the small category of affine subvarieties of $\mathbb{C}^n$, as fully detailed in [32, pg. 116-121]. This would lend itself neatly to sheaves that descend only on the Zariski open subsets. Unfortunately, it is this exact property that consigns us to failure on many fronts, like developing cohomologies.

In general, a cohomology theory should produce diverse nontrivial invariants of a given sheaf or space, helping us distinguish subtly different objects and identify the properties to blame. Unfortunately, the Zariski topology is far too coarse for such a task: for any complex variety X and constant sheaf $\mathscr{F}$[1], for instance, we find that $\mathrm{H}^i(X, \mathscr{F}) = 0$ for all $i > 0$ [53, Tag 03N3]. This is because for any irreducible variety, all Zariski open subsets meet, so the sheaf's restriction maps are all surjective. This makes all higher Čech cohomology groups vanish, so our Zariski cohomology will too [53, Tag 03N3].

It is not obvious how to directly add more open subsets to a variety without compromising its structure. The accepted alternative is to use a Grothendieck topology whose morphisms are not necessarily monic, such as the *étale topology*, the *fidèlement plat et quasicompact (fpqc) topology*, or the *fidèlement plat de presentation finie (fppf) topology*. Intuitively, an *étale morphism* should be thought of as a covering space on its (open) image [53, Tag 0257] [53, Tag 039N], which in achieving this goal becomes somewhat of a local isomorphism. More precisely, $f : X \to Y$ is étale if and only if every point $x \in X$ has a neighborhood $\mathrm{Spec}(U) \subset X$ and $f(x)$ has a neighborhood $\mathrm{Spec}(V)$ such that $f(\mathrm{Spec}(U)) \subseteq \mathrm{Spec}(V)$ and the induced $V \to U$ is smooth and of relative dimension equal to zero [53, Tag 00U0] [53, Tag 02GI]. Commelin in [7] explains how these are local isomorphisms, which is where we will direct the interested reader. The topology generated by surjective families of étale morphisms defines the étale topology, which carries a number of functorial and cohomologically satisfying properties that can be found in more detail in [53, Tag 024J] [53, Tag 03N1].

It is not hard to then define the fppf and fpqc topologies, a favorable consequence of unoriginal naming. Indeed, the fppf topology is simply generated by the faithfully flat morphisms locally of finite presentation [53, Tag 021M] and likewise for the faithfully flat and quasicompact morphisms with the fpqc topology [53, Tag 022B]. It is shown in [53, Tag 022C] that, in fact, every single covering we have discussed (Zariski, fppf, étale) are special cases of fpqc coverings, though it also suffers set-theoretic issues that the others do not. We will elect to use it regardless, though this choice can be changed at any point if it suits application.

In [53], sites with these topologies are usually taken to be small, ie. some elected set of schemes with corresponding coverings. We will either ignore this and use our other formalisms, or assume it to be implicit.

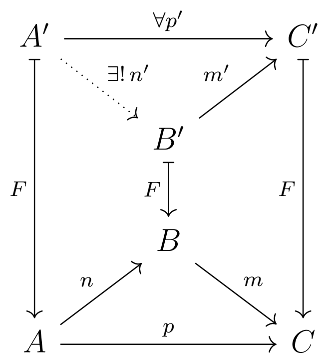
[1] This means $\mathscr{F}(U)$ is the set of locally constant maps to some fixed set for all U [53, Tag 006W].

2.2 Fibred Categories and Descent Data

There are several ways to augment a moduli problem's presheaf of families $\mathscr{F}$ so it contains all the equivalences between families explicitly. One is to convert $\mathscr{F}$ into a 'groupoid-valued sheaf' $\mathscr{F} : \mathscr{C}^{op} \to \mathbf{Grpd}$, where each groupoid represents a class of families with equivalences as morphisms. We will instead elect to switch to a *fibred category*, which may be shown to be equivalent for our purposes [53, Tag 0049]. We will in fact eventually return to the first approach again much later.

What is a fibred category? In essence, given a category $\mathscr{C}$, we are interested in categories $\mathscr{S} \to \mathscr{C}$ over $\mathscr{C}$, where for any $f : A \to B \in \mathrm{Mor}(\mathscr{C})$ and $x' \in \mathscr{S}$ mapping to B we can produce a 'fibre product' $A \times_B x'$ in $\mathscr{S}$ [53, Tag 02XJ], which for our purposes will correspond to a pullback of a family x' over B to one over A. To do this, we will have to define the kind of morphism we expect $f' : A \times_B x' \to x'$ to be in general, for which we follow Giraud [19, ch. 1].

Let $F : \mathscr{S} \to \mathscr{C}$ be a functor. Furthermore, let $m' \in \mathrm{Mor}(\mathscr{S})$, $m = F(m')$ and $n \in \mathrm{Mor}(\mathscr{C})$ such that $\mathrm{cod}(n) = \mathrm{dom}(m)$ so we may define $p = m \circ n$. Now, consider the logical predicate $c(m', n)$ defined to be true exactly when for all $p' \in \mathrm{Mor}(\mathscr{S})$ such that $F(p') = p$ and $\mathrm{cod}(p') = \mathrm{cod}(m')$ there is a unique $n' \in \mathrm{Mor}(\mathscr{S})$ such that $m' \circ n' = p'$ and $F(n') = n$. This can be represented diagrammatically for $p' : A' \to C'$ and $m' : B' \to C'$:



Implicit in the above picture is the fact that F maps each morphism above to the respective ones below.

Definition 2.4. [19, pg. 1-2] Let $F : \mathscr{S} \to \mathscr{C}$ be a functor with $m' \in \mathrm{Mor}(\mathscr{S})$ a morphism.

1. m' is *cartesian* (relative to $\mathscr{C}$) if $c(m', 1_{\mathrm{dom}(F(m'))})$.

2. m' is *strongly cartesian* (relative to $\mathscr{C}$) if $c(m', n)$ for all morphisms n such that $\text{dom}(F(m)) = \text{cod}(n)$.

Note that [19] uses the terminology *hypercartésien*, which we convert to *strongly cartesian* to follow the more modern [53].

Being strongly cartesian is, in essence, declaring that any factorization of a morphism p through $F(m')$ should induce a unique factorization of all its fibres p' through m'.

Definition 2.5. [53, Tag 02XM] Let $F : \mathscr{S} \to \mathscr{C}$ be a functor. Say that $\mathscr{S}$ is *fibred over* $\mathscr{C}$ if, for every morphism $m \in \text{Mor}(\mathscr{C})$ and every $x' \in \mathscr{S}$ such that $F(x') = \text{cod}(m)$, there exists a strongly cartesian $m' : y' \to x' \in \text{Mor}(\mathscr{S})$ such that $F(m') = m$.

[19, pg. 2] denotes this phenomenon by stating F to be *fibrant*, including the notion of F being *prefibrant* should the above definition be stated with the term 'cartesian' instead of 'strongly cartesian'. We will avoid this terminology as it appears to be far less recent and may only serve to confuse when we discuss fibrations later.

We can now see how we are able to generate fibre products across the functor, as per the following informal picture:

$$\begin{array}{ccc} V \times_U x' & \xrightarrow{m'} & x' \\ {\scriptstyle F}\Downarrow & & \Downarrow{\scriptstyle F} \\ V & \xrightarrow{m} & U \end{array}$$

Indeed, the definition of a fibred category begins with the data of m and x', the lower-right corner of the 'commutative diagram', and demands the existence of a universal upper-left corner, owing to m' being strongly cartesian. It is this behavior that attracts us to fibred categories to describe moduli problems, where in the above picture we are pulling back a family x' over U to one over V.

We will from now on say that $\mathscr{S}$ is a *category over* $\mathscr{C}$ when there is an implicit functor $F : \mathscr{S} \to \mathscr{C}$. Furthermore, for any $U \in \mathscr{C}$, we will write $\mathscr{S}_U$ for the *fibre category over* U, consisting of all objects $x \in \mathscr{S}$ such that $F(x) = U$ and all morphisms $f \in \text{Mor}(\mathscr{S})$ such that $F(f) = 1_U$. Clearly this is a category, as it contains identities and is closed under composition. Note that the strongly cartesian morphisms in $\mathscr{S}_U$ are precisely all its isomorphisms.

It is possible to, for every $f : U \to V$ in $\mathscr{C}$ and every $x' \in \text{Ob}(\mathscr{S}_V)$, make a choice of a strongly cartesian morphism $f^*x' \to x' \in \text{Mor}(\mathscr{S})$ sent by F to f. Assuming $\mathscr{C}$ to be locally small, we can use the axiom of choice to do this for every $f \in \mathrm{Hom}_{\mathscr{C}}(U, V)$ [53, Tag 02XJ]. This can then be done for all such hom-sets simultaneously[2]. Note that the choices we make will be unique up to unique isomorphism: for two strongly cartesian morphisms $f' : a \to b$ and $g' : c \to b$ such that $F(f') = F(g')$, we must have a unique isomorphism $a \to c$ that makes f' factor through g'.

It is this construction that will let us formulate a functorial method to, given any $f : U \to V$, pull back the fibre category $\mathscr{S}_V$ to $\mathscr{S}_{\mathscr{U}}$. More specifically, we can design a functor $f^* : \mathscr{S}_V \to \mathscr{S}_{\mathscr{U}}$ such that f^*x' is the object we chose in our choice of morphism $f^*x' \to x'$ for each $x' \in \mathrm{Ob}(\mathscr{S}_V)$. This merely requires us to define $f^*\phi$ for a morphism $\phi : y' \to x' \in \mathrm{Mor}(\mathscr{S}_V)$, which can be done naturally by noting the existence of a unique morphism $f^*y' \to f^*x'$ such that the diagram

$$\begin{array}{ccc} f^*y' & \longrightarrow & f^*x' \\ \downarrow & & \downarrow \\ y' & \xrightarrow{\phi} & x' \end{array}$$

commutes, owing to $f^*x' \to x'$ and $f^*y' \to y'$ being strongly cartesian [53, Tag 02XJ]. This defines our pullback functor f^*, a notion that will replace the pullback morphisms defined by a moduli problem's presheaf of families $\mathscr{F}$.

We now make all the discussed notions so far concrete.

Definition 2.6. [53, Tag 02XN] Let $F : \mathscr{S} \to \mathscr{C}$ be a fibred category.

1. A *choice of pullbacks*, or *cleavage*, is a choice of strongly cartesian morphism $f^*x' \to x'$ for every $f : U \to V$ in $\mathscr{C}$ and $x' \in \mathrm{Ob}(\mathscr{S}_V)$.

2. For a given choice of pullbacks and some $f : U \to V$ in $\mathscr{C}$, the *pullback functor* $f^* : \mathscr{S}_V \to \mathscr{S}_U$ is the functor defined as above.

It is somewhat clear that for any $f : U \to V$ and $g : V \to W$, we have a unique natural isomorphism $\alpha_{f,g} : (g \circ f)^* \to f^* \circ g^*$ such that for every $y \in \mathscr{S}_W$ the diagram

$$\begin{array}{ccc} (f^* \circ g^*)y & \longrightarrow & f^*y \\ {\scriptstyle (\alpha_{f,g})_y}\uparrow & & \downarrow \\ (g \circ f)^*y & \longrightarrow & y \end{array}$$

commutes [53, Tag 02XO] by the strongly cartesian property and the fact that strongly cartesian morphisms are closed under composition [53, Tag 02XL]. We can perhaps

[2] We have left this as a rather informal statement, since proving its feasiblity is more a question of foundations.

now see, as noted in [64, pg. 71], how a fibred category should be thought of as a presheaf of categories; the set $\mathscr{F}(U)$ has been replaced with the fibre $\mathscr{S}_U$ and morphisms f^* have become functors compatible with composition.

2.2.1 Categories Fibred in Groupoids

We are interested in assigning to each $U \in \mathscr{C}$ a groupoid, hence we would like to our fibres to contain only isomorphisms. We thus achieve the following definition, a reworking of [53, Tag 003V]:

Definition 2.7. Let $\mathscr{S}$ be a category over $\mathscr{C}$. Then $\mathscr{S}$ is *fibred in groupoids over* $\mathscr{C}$ if it is fibred over $\mathscr{C}$ and all of its fibre categories are groupoids.

[53] chooses an alternative definition before proving ours to be equivalent, which we ignore to maintain intuition. An important detail, however, is that we must declare $\mathscr{S}$ to be fibred over $\mathscr{C}$; having every fibre category be a groupoid does not imply this. Indeed, given such a category, we could artificially add 'copies' of morphisms not in any fibre, which leaves the fibres unaltered but violates uniqueness of strongly cartesian pullbacks [53, Tag 003U].

It is not hard to define a morphism between categories fibred over some $\mathscr{C}$, being a functor that commutes with the morphisms down to $\mathscr{C}$ which preserves strongly cartesian morphisms [53, Tag 02XP]. The same thus holds for categories fibred in groupoids, giving us a category of each for some fixed $\mathscr{C}$.

2.2.2 From Presheaves to Fibred Categories

We mentioned in passing at the start of this section that a contravariant functor $F : \mathscr{C}^{op} \to \mathbf{Grpd}$ to the category of groupoids was an equivalent formulation of categories fibred in groupoids and vice versa. We now make this correspondence explicit.

Proposition 2.1. [53, Tag 02XV] Let $F : \mathscr{C}^{op} \to \mathbf{Cat}$ be a functor to the category of small categories. Define the category S_F whose objects are pairs (U, x) for $U \in \mathscr{C}, x \in F(U)$ and morphisms $(U, x) \to (V, y)$ being pairs of morphisms (f, ϕ) such that $f : U \to V$ is in $\mathscr{C}$ and $\phi : x \to F(f)(y)$ is in $F(U)$. Then S_F is a category fibred over $\mathscr{C}$ via the forgetful functor $p_F : S_F \to \mathscr{C}$ sending (U, x) to U and (f, ϕ) to f.

Proof. We must first show S_F to be a category at all. Our composition of morphisms will be defined by the commuting diagram

$$(U,x) \xrightarrow{(f,\phi)} (V,y) \xrightarrow{(g,\psi)} (W,z), \qquad (U,x) \xrightarrow{(g\circ f,\, F(f)(\psi)\circ\phi)} (W,z)$$

It is clear in this case that composition is associative, deriving merely from F being a functor and both $\mathscr{C}$ and $F(U)$ being categories for all $U \in \mathscr{C}$. The identity of (U,x) is of course $(1_U, 1_x)$, indicating S_F to be a category as expected.

To show it is fibred, we must have a strongly cartesian lift of each morphism $f : U \to V$ in $\mathscr{C}$ for each $(V,x) \in S_F$. Take the lift $(f, 1_{F(f)(x)}) : (U, F(f)(x)) \to (V,x)$. Proving this to be strongly cartesian requires any h in the diagram on the left to determine a unique (h,μ) on the right:

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ {\scriptstyle h}\uparrow & \nearrow_{g} & \\ W & & \end{array} \qquad\qquad \begin{array}{ccc} (U, F(f)(x)) & \xrightarrow{(f,1_{F(f)(x)})} & (V,x) \\ {\scriptstyle (h,\mu)}\uparrow & \nearrow_{(g,\psi)} & \\ (W,z) & & \end{array}$$

We can clearly just take $\mu = \psi$. This works since $f \circ h = g$ and so $F(g)(x) = F(h)(F(f)(x))$. Evidently, there is no other lift making the diagram on the right commute, so we are done. $\square$

A dual construction of a functor $\mathscr{C}^{op} \to$ **Cat** from a fibred category is presented in [53, Tag 004A]. However, we will only make use of the first. Of course, this construction takes functors into **Grpd** to categories fibred in groupoids. The reader should also maintain caution that this correspondence can only be made for fibred categories whose fibre categories are small.

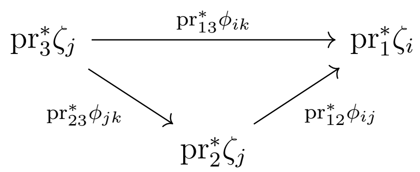
2.2.3 Descent on Fibred Categories

So far, we have augmented our original concept of a presheaf so it is valued in groupoids rather than sets. This sets the stage for our new presheaf of families $\mathscr{F}$, now simply a functor $F : \mathscr{S} \to \mathscr{C}$ from the category of families to the category of spaces, sending each family to the space it is parameterized by. Each fibre $\mathscr{S}_U$ will be a groupoid of families parameterized by U with isomorphisms that restrict to 1_U, which by our definitions of a category fibred in groupoids can be 'pulled back' across maps between spaces.

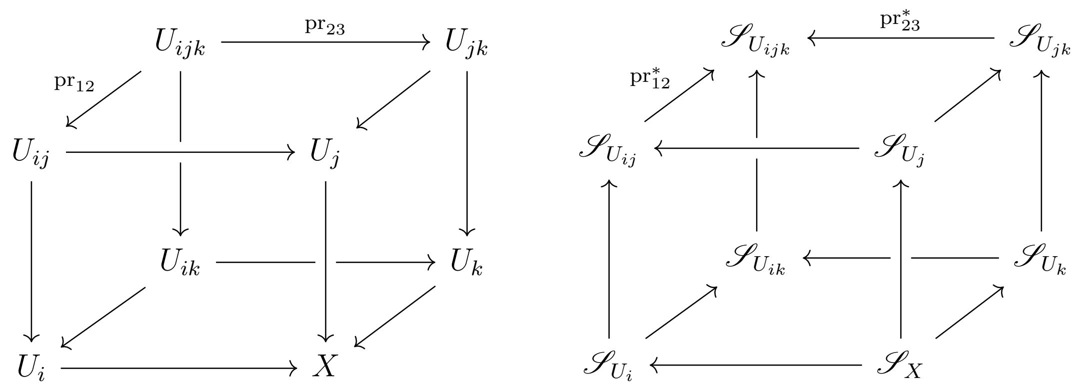
This perhaps sounds sufficient to replace our original presheaf of families $\mathscr{F}$, but there is still more we need: in order to ensure we pay respect to the topology involved, our so-called 'presheaf of groupoids' needs to become a 'sheaf of groupoids' [64, pg. 71]. For this, as we did for set-valued sheaves, we must develop a suitable concept of descent.

From now on we will assume $\mathscr{C}$ to be a site (with pullbacks) and $F : \mathscr{S} \to \mathscr{C}$ to be a category fibred in groupoids over $\mathscr{C}$. We will also in general assume some choice of pullbacks has been made. For a covering $\{U_i \to X \mid i \in I\}$ of some $X \in \mathscr{C}$, denote $U_{ij} = U_i \times_X U_j$ and $U_{ijk} = U_i \times_X U_j \times_X U_k$, where bracketing is of course not important. Furthermore, denote pr_a or pr_{ab} for projection onto the a^{th} or ab^{tb} coordinates, respectively.

Definition 2.8. [64, pg. 71-72] Let $S = \{U_i \to X \mid i \in I\}$ be a covering of X in a site $\mathscr{C}$ and $\mathscr{S} \to \mathscr{C}$ a fibred category. An *object with descent data* $(\{\zeta_i\}, \{\phi_{ij}\})$ on S is a collection of objects $\zeta_i \in \mathscr{S}_{U_i}$ together with isomorphisms $\phi_{ij} : \mathrm{pr}_2^*\zeta_j \cong \mathrm{pr}_1^*\zeta_i$ in $\mathscr{S}_{U_{ij}}$, such that for any $i, j, k \in I$ we have the commutative diagram in $\mathscr{S}_{U_{ijk}}$



The last diagram in the above definition is called the *cocycle condition* and the morphisms ϕ_{ij} are the *transition isomorphisms*. The commutative[3] diagrams



inspired by [64, pg. 72] should help visualization, where we should see morphisms in the left cube as 'inclusions' of the 'intersections' U_{ij} and U_{ijk}. The projection maps $\mathrm{pr}_a : U_{abc} \to U_a$ are then the compositions $U_{abc} \to U_{ab} \to U_a$ in the diagram. This

[3]All paths in the rightmost cube are the same up to a unique natural isomorphism, so we may behave as though it is commutative.

can be lifted to the diagram on the right, moving to fibre categories and pullback functors. The cocycle condition then takes place entirely in $\mathscr{S}_{U_{abc}}$ and the transition isomorphisms in each $\mathscr{S}_{U_{ab}}$.

This is essentially the same as a matching family for a sheaf of sets on a site, where equality is weakened to isomorphism. To this end, we should in turn expect some notion of an amalgamation, which will be a kind of mapping from $\mathscr{S}_X$ to the collection of all objects with descent data on coverings of X. To respect the categorical structure of this fibre category, we should apply a suitable category theoretic structure to these objects and produce a corresponding functor. Thankfully, Vistoli [64, pg. 72] notes there to really only be one natural such category.

Definition 2.9. [64, pg. 72] A morphism between objects with descent data $(\{\zeta_i\}, \{\phi_{ij}\})$ and $(\{\eta_i\}, \{\psi_{ij}\})$ is a collection of morphisms $\alpha_i : \zeta_i \to \eta_i$ in $\mathscr{S}_{U_i}$ such that for each $i, j \in I$ the diagram

$$\begin{array}{ccc} \mathrm{pr}_2^*\zeta_j & \xrightarrow{\mathrm{pr}_2^*\alpha_j} & \mathrm{pr}_2^*\eta_j \\ {\scriptstyle \phi_{ij}}\downarrow & & \downarrow{\scriptstyle \psi_{ij}} \\ \mathrm{pr}_1^*\zeta_i & \xrightarrow{\mathrm{pr}_1^*\alpha_i} & \mathrm{pr}_1^*\eta_i \end{array}$$

commutes in $\mathscr{S}_{U_{ij}}$. The category of objects with descent data on a covering S, written $DD(S)$ [53, Tag 026B], is defined accordingly.

We now wish to construct a functor $\mathscr{S}_X \to DD(S)$ for any covering $S = \{f_i : U_i \to X\}$ of X in our site $\mathscr{C}$ that should somehow correspond to amalgamations. This is not too difficult: any object $x' \in \mathscr{S}_X$ can be pulled back repeatedly to get the object with descent data $(\{f_i^* x'\}, \{\phi_{ij}\})$ where ϕ_{ij} are the unique natural isomorphisms arising from the projection maps, which we may treat as identities [64, pg. 72]. This action defines our functor, which we informally call the *amalgamation functor*. As per [64, pg. 76] and [53, Tag 026E], we will call an element of $DD(S)$ *effective* if is in the essential image of this functor, meaning it can be 'amalgamated' into an element of $\mathscr{S}_X$ up to isomorphism. This replaces the sheaf condition we had previously explored.

It should be noted in passing that, in fact, any object with descent data can be pulled back in a similar manner:

Lemma 2.1. [53, Tag 02ZD] Let $S = \{U_i \to X \mid i \in I\}$, $T = \{V_i \to Y \mid i \in J\}$ be collections of morphisms in the site $\mathscr{C}$ such that $V_{ij}, V_{ijk}, U_{ij}, U_{ijk}$ all exist for all i, j, k. Let $\alpha : I \to J$ be a mapping, $h : X \to Y$ a morphism and $g_i : U_i \to V_{\alpha(i)}$ a collection of morphisms commuting with h, S and T.

For any object with descent data $(Y_j, \phi_{jj'}) \in DD(T)$, the object

$$(g_i^*(Y_{\alpha(i)}), (g_i \times g_{i'})^*(\phi_{\alpha(i)\alpha(i')}))$$

is in $DD(S)$. This construction forms a functor $h^* : DD(T) \to DD(S)$. Moreover, if $\alpha' : I \to J$ is another mapping and $g_i : U_i \to V_{\alpha'(i)}$ is another collection of morphisms commuting with h, S and T then the induced functors will be canonically isomorphic.

We omit the proof of this lemma, though it should largely be self-evident. Note that the amalgamation functor is just a pullback of $(x, 1_x) \in DD(\{1_X\})$ to the covering S, where we are implicitly extending $DD(\cdot)$ to collections of morphisms that are not coverings yet are closed under the necessary pullbacks.

2.3 Stacks

Finally, after all our preparation, the major object of study in this chapter reveals itself.

Definition 2.10. [64, pg. 75] A fibred category $\mathscr{S}$ over a site $\mathscr{C}$ is a *stack* if for every $X \in \mathscr{C}$ and covering S of X, the amalgamation functor $\mathscr{S}_X \to DD(S)$ is an equivalence of categories.

Definition 2.11. [53, Tag 02ZJ] A fibred category $\mathscr{S}$ over a site $\mathscr{C}$ is a *stack in groupoids* if it is a stack and is fibred in groupoids.

Our definition seems to implicitly rely on a choice of cleavage. However, [44, pg. 96-97] proves this definition to in fact be invariant under such choice, which we will simply take for granted.

We should take a moment now and try to understand what it is we have just defined. A stack is now, in essence, a presheaf of categories where every so-called matching family can be amalgamated, ie. one where all descent data is effective. This should lead us to see it as a kind of 'sheaf in categories'. A stack in groupoids is similarly a sheaf in groupoids with regards to our site $\mathscr{C}$.

An alternative formulation of stacks involves, for a fibred category $\mathscr{S} \to \mathscr{C}$ and $x, y \in \mathscr{S}_U$, the presheaf $\underline{\mathrm{Hom}}(x, y) : (\mathscr{C}/U)^{op} \to \mathbf{Set}$ defined by $(f : V \to U) \mapsto \mathrm{Hom}_{\mathscr{S}_{\mathscr{C}}}(f^*(x), f^*(y))$ and for any $g : W \to V$ over U a restriction map

$$\mathrm{Hom}_{\mathscr{S}_V}(f^*(x), f^*(y)) \xrightarrow{g^*} \mathrm{Hom}_{\mathscr{S}_W}(g^* f^*(x), g^* f^*(y)) \cong \mathrm{Hom}_{\mathscr{S}_W}((f \circ g)^*(x), (f \circ g)^*(y))$$

defined by g^* and the isomorphism $\alpha_{g,f}$ [44, pg. 97]. This is clearly a valid presheaf as it respects composition and identities by definition. Moreover, different cleavages will give canonically isomorphic presheaves [44, pg. 98].

We have the following fact from [44, pg. 98]:

Proposition 2.2. For any fibred category $\mathscr{S}$ over $\mathscr{C}$, $\underline{\mathrm{Hom}}(x, y)$ is a sheaf for all $U \in \mathscr{C}$ and $x, y \in \mathscr{S}_U$ if and only if the amalgamation functor is fully faithful for all coverings S of all $U \in \mathscr{C}$.

Proof. Let $S = \{f_i : U_i \to X\}$ be a covering and $g_i : U_i \times_X U_i \to X$ be the unique map along each pullback. Then the pullback square is sent by the presheaf to the rather familiar diagram

$$\mathrm{Hom}_{\mathscr{S}_X}(x, y) \longrightarrow \mathrm{Hom}_{\mathscr{S}_{U_i}}(f^*(x), f^*(y)) \rightrightarrows \mathrm{Hom}_{\mathscr{S}_{U_i \times_X U_i}}(g^*(x), g^*(y))$$

Consider a collection of morphisms $\phi_i \in \underline{\mathrm{Hom}}(x, y)(U_i)$. Along with this, denote the images of $f^*(x)$ and $f^*(y)$ under the amalgamation functor as X and Y respectively. The morphisms ϕ_i have the same image in the two maps above exactly when they form a morphism $X \to Y$ in $DD(S)$, which we leave the reader to quickly verify. This tells us that the sheaf condition above is satisfied exactly when every morphism between effective objects with descent data is a morphism of the source objects, meaning the amalgamation functor $\mathscr{C} \to DD(S)$ is fully faithful.

This is enough to show the sheaf condition for all x, y, X implies full faithfulness of the amalgamation functor for every covering S of X. To see the converse, let $x, y \in \mathscr{S}_X$, $U_i \to X$ and $f : V \to X$ be objects in $\mathscr{C}/X$ and $\{U_i \to V\}$ a covering in $\mathscr{C}/X$. Note that the sheaf axiom for $\underline{\mathrm{Hom}}(x, y)$ with regards to this cover is just the sheaf condition for $\underline{\mathrm{Hom}}(f^*(x), f^*(y))$ with regards to the cover $\{U_i \to V\}$ in $\mathscr{C}/V$. By our above reasoning, this is just full faithfulness of the amalgamation functor $\mathscr{C} \to DD(\{U_i \to V\})$. □

Hence, another definition for a stack is a fibred category $\mathscr{S}$ over $\mathscr{C}$ where the presheaf $\underline{\mathrm{Hom}}(x, y)$ is a sheaf for all $U \in \mathscr{C}$, $x, y \in \mathscr{S}_{\mathscr{C}}$ and all descent data is effective. This is in fact the definition used in [53, Tag 026F] and is often easier to work with.

We now have the basic object upon which all other theory in this chapter shall be built. Indeed, we should now begin to try and transport technology into the context of stack theory and see how effective it really is. It is unfortunate that all the machinery of stacks is so vast and deep that we will have little time to fully explore it here. Instead, we will give an overview with many proofs omitted and refer the avid reader to [53], [64] and [44].

Our first order of business is to observe that, under certain conditions, a presheaf $P : \mathscr{C}^{op} \to \mathbf{Set}$ may be converted into a stack. Following [44, pg. 97], we define the fibred category P^{fib} whose objects are pairs (U, x) such that $x \in P(U)$ and morphisms $(U, x) \to (V, y)$ being maps $f : U \to V$ such that $P(f)(y) = x$. To make this fibred, we add a functor $F : P^{fib} \to \mathscr{C}$ sending (U, x) to U. In this context, if $\mathscr{C}$ is now a site, we have for any sieve $S = \{f_i : U_i \to X \mid i \in I\}$ over $X \in \mathscr{C}$ that the objects of $DD(S)$ take the form $\{(U_i, y_i) \mid \mathrm{pr}_2^* y_j = \mathrm{pr}_1^* y_i \in P(U_i \times_X U_j)\}$ where the transition isomorphisms are clearly equalities. These are just matching families, so our stack condition is now exactly the sheaf condition and a presheaf is naturally a stack exactly when it is a sheaf. It is the $\cdot^{fib}$ construction that shows the relationship between presheaves and sheaves to be identical to that between fibred categories and stacks.

To formalize the breadth of this construction, we note that stacks over a site $\mathscr{C}$ can be easily formed into a category $\mathbf{St}(\mathscr{C})$ by taking the full subcategory of fibred categories over $\mathscr{C}$ consisting of stacks. This then has a full subcategory of stacks in groupoids. In the context of the $\cdot^{fib}$ construction, however, we will take interest in an even more narrow subcategory.

Definition 2.12. [44, pg. 90] A *category fibred in sets* is a fibred category where each fibre is both small and discrete.

We may form a subcategory of the fibred categories over $\mathscr{C}$ containing only these objects. It is evident that, for any presheaf P, P^{fib} will fall into this category. This gives the following proposition, which we present without proof:

Proposition 2.3. [44, pg. 91] The functor from presheaves over $\mathscr{C}$ to fibred categories in sets over $\mathscr{C}$, sending P to P^{fib}, is an equivalence of categories.

The proof requires we generalize the Yoneda lemma to the context of fibred categories [44, pg. 89], a useful result that we lack the time for.

This lets us immediately construct a number of trivial stacks from what we already know in sheaf theory. For instance, any object X in a site $\mathscr{C}$ can be made into a unique presheaf $\mathrm{Hom}(-, X)$ by the Yoneda embedding. In many contexts, this is in fact a sheaf, such as in the obvious setting of a topological site with open covers [32, pg. 124] or in the fpqc topology over some site of schemes [53, Tag 023Q]. We call such a topology a *subcanonical topology* [44, pg. 100]. Hence, the $\cdot^{fib}$ construction will often give us a fully faithful embedding of sheaves into stacks, so we can convert any $X \in \mathscr{C}$ into a unique stack. This will be a stack in sets, ie. one which is also fibred in sets. We will therefore often refer to a scheme as though it is a stack, for simpler notation.

This procedure suggests yet another natural way to see a stack[4]. Indeed, in the case of a stack $\mathscr{X}$ that is a scheme, each fibre $\mathscr{X}_U$ is the discrete category of homomorphisms from U to X. Each of these is a 'generalized U-point', much like the Spec(k)-points being points in a k-scheme or S^1-points being loops. Further, the pullback functors tell us, given a morphism $V \to U$, how to extract a V-point from within any U-point. These functors not being equivalences imply connections between generalized points - given $x, y \in \mathscr{X}_{\mathbb{A}^1_k}$ and a morphism $f : \mathrm{Spec}(k) \to \mathbb{A}^1_k$ identifying the point 0, the equation $f^*(x) = f^*(y)$ intuitively says 'the lines x, y in $\mathscr{X}$ meet at their 0's'. The sheaf condition tells us that some generalized points agreeing on their maximal overlaps, or the fibre product, represent one unique larger point.

Inspired by this special case of a stack, we may try to see all stacks in the same way, namely an entity containing a collection of generalized U-points for each $U \in \mathscr{C}$ that can be pulled back along maps $V \to U$ and glued together uniquely, should they correspond to a covering of another type of point. So far, however, these conditions only describe a sheaf. What differentiates a stack is that many of the generalized points may have morphisms between them. This, especially in the case of a stack in groupoids, suggests the image of a shape where one 'actual point' may appear in several places. Pulling back along morphisms must now respect this, which we gain from the fibred category model. This image is exactly what we sought out for an augmentation of a moduli space.

We can now try to convert a moduli problem $A/\sim$ into a stack, which we will call a *moduli stack*. Take a site of schemes $\mathscr{C}$ with a reasonable topology, such as the étale or fppf topologies. Our chosen concept of a family will let us develop a category of all families $\mathscr{S}$, whose isomorphisms respect our original axioms. We can then easily construct a functor $F : \mathscr{S} \to \mathscr{C}$ sending each family to the scheme parameterizing it and morphisms to their restrictions on the parameterizing varieties. This is clearly fibred if pullbacks of families exist and furthermore must be a stack, as we are invariably able to glue together families that agree on overlaps, provided we have chosen a topology well.

Whether this will be a stack in groupoids will depend on how we defined our families. We could of course simply define $\mathscr{S}$ such that it only contains those morphisms that correspond to pullbacks and equivalences under $\sim$, after which we will trivially have groupoid fibres. The reader should note that this may be sometimes artificial

[4] The intuition I present here is derivative of a perspective on sheaves described in [41].

if the families have a natural structure entirely of their own rather than being constructed from scratch, where morphisms that descend to identities on parameterizing families may not be isomorphisms, or further that equivalences under $\sim$ may not be isomorphisms at all. For now, this is simply something to keep in mind for the future.

The skeptical reader may be further inclined to ask why our original presheaf of equivalence classes of families was not simply given a sheaf condition. The problem is that we may be directly interested in the morphisms between families themselves, so losing this information would contradict our motivating question. In our very first example of a moduli problem, the set of circles in $\mathbb{R}^3$ up to rotation and translation, all the equivalences reflect the fact that these circles are not intrinsic, but necessarily set within a larger space in many possible symmetric ways. The moduli stack of these, whose points are such circles, should have symmetries of its own derived from these ones; a symmetry in this case would be a rotation or translation of each of its points in a smoothly varying manner, ie. equivalences of families. In some sense, this stack is able to, at each of its points, vary in $\mathbb{R}^3$ itself. Hence, the stack records all of its equivalent 'symmetric' configurations by the isomorphisms in its fibre categories, evidently required to be compatible with all pullback functors.

An interesting example of a moduli stack in [44, pg. 33] comes from the presheaf in groupoids $\mathscr{M}_g$ for $g \geq 2$, where $\mathscr{M}_g(S)$ is the category whose objects are smooth proper maps $\pi : C \to S$ whose geometric fibres are connected genus g curves, with isomorphisms over S. This naturally becomes a category fibred in groupoids, which is proven in [44, pg. 33-34] to be a stack in groupoids over the fppf topology. Some fine-tuning is required to make this work for elliptic curves, ie. $\mathscr{M}_1$, for which we refer the reader to [44, pg. 97].

Another example comes from the quasi-coherent sheaves over a scheme X. Given a quasi-coherent sheaf P over the site of Zariski open subsets of a scheme X, we gain a sheaf on an fppf site $\mathbf{Sch}/X$ by setting $P(f : T \to X) = \Gamma(T, f^*P)$ [44, pg. 30]. With this in mind, we define $\mathrm{Qcoh}(X)$ to be the category of sheaves P over the fppf site $\mathbf{Sch}/X$ where for any $f : T \to X$, the restriction P_T of P to the Zariski topology on T is quasi-coherent and for all $g : T \to T'$, $g^*P_{T'} \cong P_T$. This, when converted to a fibred category, is also in fact a stack [44, pg. 97].

Given our examples of stacks so far, we may perhaps be confused as to how one discerns properties of the stack itself. Is it quasi-compact? Finitely presented? Smooth? As with all of algebraic geometry, such properties are often best phrased as attributes of morphisms rather than objects. By doing this, we will be able to 'probe' the stack with schemes. Indeed, consider the following picture, where $\mathscr{X}$ and $\mathscr{Y}$ are stacks and A is a scheme:

$$\begin{array}{ccc} & & A \\ & & \big\downarrow f \\ \mathscr{Y} & \xrightarrow{g} & \mathscr{X} \end{array}$$

We would like to observe some property of the morphism g. In essence, f identifies a region of $\mathscr{X}$ that is sufficiently well-behaved to constitute a scheme itself, which we may pull out of the stack to study on its own. To do so, we must have some kind of fibre product $\mathscr{Y} \times_{\mathscr{X}} A$ which is a scheme, so we may analyze it classically. Before we proceed however, we need to define what such a fibre product in this context actually is.

Definition 2.13. [44, pg. 106] Let $f : \mathscr{A} \to \mathscr{S} \leftarrow \mathscr{B} : g$ be a diagram of fibred categories over $\mathscr{C}$. Define the fibre category $\mathscr{A} \times_{\mathscr{S}} \mathscr{B}$ whose objects are 4-tuples (U, a, b, μ), where $U \in \mathscr{C}$, $a \in \mathscr{A}_U$, $b \in \mathscr{B}_U$ and $\mu : f(a) \cong g(b)$ is an isomorphism in $\mathscr{S}_U$. The morphisms $(U, a, b, \mu) \to (U', a', b', \mu')$ are pairs of morphisms $\psi : a \to a'$ in $\mathscr{A}$ and $\chi : b \to b'$ in $\mathscr{B}$ such that the diagram in $\mathscr{S}$

$$\begin{array}{ccc} f(a) & \xrightarrow{\mu} & g(b) \\ \big\downarrow f(\psi) & & \big\downarrow g(\chi) \\ f(a') & \xrightarrow{\mu'} & g(b') \end{array}$$

commutes.

This is rather clearly a fibred category over $\mathscr{C}$ and satisfies the universal property of a fibre product. We also note that if $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{S}$ are all stacks, then $\mathscr{A} \times_{\mathscr{S}} \mathscr{B}$ will be one as well: an object with descent data in this category descends to objects on $\mathscr{A}$ and $\mathscr{B}$ agreeing on $\mathscr{S}$ and is effective if and only if it is as such on these three categories as well. Furthermore, we note $\mathscr{A} \times_{\mathscr{S}} \mathscr{B}$ to be fibred in groupoids if its three defining fibred categories are themselves [44, pg. 106].

With this in mind, we may now address our original problem with a new definition. We assume in this context our site $\mathscr{C}$ to be one of schemes.

Definition 2.14. [53, Tag 02ZQ] A stack is *representable* if it is equivalent to the stack of some scheme. A morphism $f : \mathscr{X} \to \mathscr{Y}$ of stacks is *representable* if for every scheme $U \in \mathscr{C}$ and every morphism $U \to \mathscr{Y}$, the stack $\mathscr{X} \times_{\mathscr{Y}} U$ is representable [44, pg. 107].

We've taken a collage of two different definitions here. Olsson in [44] weakens representability from schemes to more general algebraic spaces, a concept we do not have the time for. [53] presents this form of representability later.

Representability will let us convert properties of schemes into properties of stacks. Note however that we will require such properties to be *stable* under our topology:

Definition 2.15. [44, pg. 39] A subcategory $\mathscr{D} \subset \mathscr{C}$ is *closed* if it contains all isomorphisms and is closed under pullbacks of morphisms, that is, $f : A \to B$ being in $\mathscr{D}$ implies $f' : A \times_B C \to C$ is too for any $C \to B$ in $\mathscr{C}$. Furthermore, a subcategory is *local on the base* if for all $f : X \to Y$ in $\mathscr{C}$ and coverings $\{U_i \to Y\}$, $f \in \mathrm{Mor}(\mathscr{D})$ if and only if all $f_i : X \times_Y U_i \to U_i$ are too. Should $\mathscr{D}$ be both closed and local on the base, we call it *stable*.

Definition 2.16. If a property P of morphisms in $\mathscr{C}$ is closed under composition and contains all isomorphisms, we say it is *stable* if the subcategory $\mathscr{C}^P \subset \mathscr{C}$ containing all objects and all morphisms satisfying P is as such.

A property that is stable is, in essence, one that respects the topology in question; a morphism has a property exactly when its restriction to various elements in a covering do as well. Examples in the étale topology, for instance, include being surjectve, universally closed or open, separated, quasi-compact, locally of finite type, flat, smooth and more [44, pg. 107].

Definition 2.17. [44, pg. 107] Let P be a stable property of morphisms of schemes, under the currently used topology. Then a morphism of stacks $f : \mathscr{X} \to \mathscr{Y}$ has property P if it is representable and for any morphism $U \to \mathscr{Y}$ from a scheme $U \in \mathscr{C}$, the morphism $\mathscr{X} \times_{\mathscr{Y}} U \to U$ has property P.

With this, a wide variety of important properties can now be demanded of morphisms between stacks.

Another way to inspect properties of an object in algebraic geometry is, in general, to look at its diagonal. We of course have a diagonal map of stacks $\Delta_{\mathscr{X}} : \mathscr{X} \to \mathscr{X} \times_{\mathscr{Y}} \mathscr{X}$ for any morphism $\mathscr{X} \to \mathscr{Y}$ since we have fibre products. In general, as we did implicitly with varieties and $\mathrm{Spec}(k)$, we will work with some category of schemes over a scheme S, so our morphism will be one of the form $\mathscr{X} \to S$.

There is a rather surprising side effect to considering properties of the diagonal:

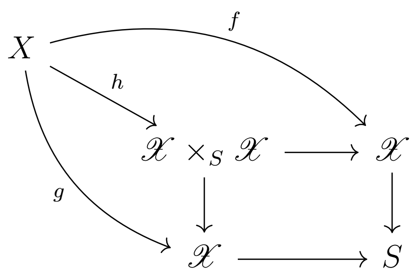
Proposition 2.4. Let $\mathscr{X}$ be a stack over a site of schemes over some scheme S. Then the diagonal $\Delta_{\mathscr{X}} : \mathscr{X} \to \mathscr{X} \times_S \mathscr{X}$ is representable if and only if every morphism $U \to \mathscr{X}$ from a scheme U over S is representable.

Proof. Our proof follows [65, pg. 11]. To begin, suppose our diagonal is representable. Let $f : X \to \mathscr{X}$ is a morphism from a scheme X over S. We must show for all other such morphisms $g : Y \to \mathscr{X}$ that $X \times_{\mathscr{X}} Y$ is a scheme over S. We gain a commutative diagram

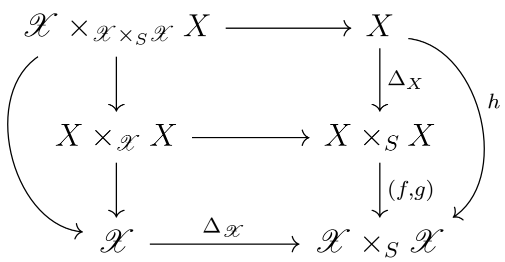
$$\begin{array}{ccc} X \times_{\mathscr{X}} Y & \longrightarrow & X \times_S Y \\ \downarrow & & \downarrow{\scriptstyle (f,g)} \\ \mathscr{X} & \xrightarrow{\Delta_{\mathscr{X}}} & \mathscr{X} \times_S \mathscr{X} \end{array}$$

where the leftmost vertical arrow is the composition along the pullback and the uppermost horizontal arrow arises from the morphism $\mathscr{X} \to S$ defining the diagonal. If we apply the definition of a fibre product of fibre categories, this square is clearly cartesian. By the diagonal being representable, we must therefore have that $X \times_{\mathscr{X}} Y$ is a scheme over S.

Conversely, suppose that every morphism $X \to \mathscr{X}$ from a scheme X over S is representable. We wish to show for every morphism $h : X \to \mathscr{X} \times_S \mathscr{X}$ the fibre product $\mathscr{X} \times_{\mathscr{X} \times_S \mathscr{X}} X$ is representable. Note that any such morphism fits into a commuting diagram of the form



where the morphisms f, g are simply compositions in the diagram. This leads us to realize $h = (f, g) \circ \Delta_X$. We can then develop a cartesian diagram



where the lower square is cartesian by our previous discussion and the outermost square is by definition. The arrow $\mathscr{X} \times_{\mathscr{X} \times_S \mathscr{X}} X \to X \times_{\mathscr{X}} X$ comes from the universality of $X \times_{\mathscr{X}} X$ as a pullback, making the diagram commute and the uppermost square cartesian. This implies

$$\mathscr{X} \times_{\mathscr{X} \times_S \mathscr{X}} X \cong (X \times_{\mathscr{X}} X) \times_{X \times_S X} X$$

Because every morphism $X \to \mathscr{X}$ is representable, $X \times_{\mathscr{X}} X$ must be a scheme over S so the right hand side of the above equation is as well. □

We now realize that being able to compare a stack to schemes, an act usually done by considering morphisms between the two with interesting properties, is truly only feasible when the stack in question has a representable diagonal morphism.

2.3.1 Artin Stacks

Our notion of a stack so far, even one of groupoids, could perhaps be better specialized to answer our original question about moduli theory. A moduli stack is supposed to contain the geometric structure of a moduli problem with a complex equivalence relation. In our earlier exploration of moduli spaces, we noted that all the pathological behaviors that rendered a moduli space nonexistent vanished if the equivalence relation was made trivial, so long as the moduli problem then remains bounded. We could therefore always expect some such simpler space to surject onto our moduli stack appropriately.

Furthermore, we should always expect the capacity to compare our stack to an assortment of schemes. It is these two requirements that will motivate our new refinement of a stack to better fit our needs.

Definition 2.18. [44, pg. 107] Given a site of schemes over S, an *Artin stack*, or *algebraic stack*, is a stack in groupoids $\mathscr{X}$ with a representable diagonal $\Delta_{\mathscr{X}} : \mathscr{X} \to \mathscr{X} \times_S \mathscr{X}$ and a scheme X over S with a smooth surjection $X \to \mathscr{X}$.

Note that the smooth surjection makes sense to discuss, given the diagonal is representable and these properties are stable under the topology we choose. In fact, any fpqc topology is admissible here: smoothness and surjectivity are both closed [53, Tag 0495] [53, Tag 00T4] and local on the base [53, Tag 02KV] [53, Tag 02VL]. We should see the scheme smoothly surjecting onto us much like an atlas of a manifold, which inspires the name *smooth atlas* [16, pg. 4].

If we require the surjection of the atlas to also be étale, we get a *Deligne-Mumford stack* [53, Tag 03YO]. This change corresponds to the difference between a submersion and local diffeomorphism [7].

We should perhaps mention that many sources such as [16] and [44] in fact demand representability to extend to objects called *algebraic spaces*, namely Deligne-Mumford stacks in sets over our site of schemes over S [53, Tag 025Y]. Indeed, [16] refers to what we call representable as *strongly representable* for this reason. While algebraic spaces are incredibly important in the general theory of stacks, presenting their theory is beyond the scope of this dissertation. We will therefore often restrict ourselves to schemes where algebraic spaces would be considered.

2.3.2 Quotient Stacks

A fruitful example of Artin stacks will come from the natural extension of a quotient space. Indeed, given our intuition of a stack in groupoids as a space where a generalized point may appear several times according to a fibre's isomorphisms, our knowledge of classical moduli spaces should raise some flags and lead us to ask when these isomorphisms are induced by some group action. More generally, we could ask when these are induced by some *groupoid object*.

Definition 2.19. [44, pg. 99] A *groupoid object* in a category $\mathscr{C}$ is a tuple $X_\bullet = (X_0, X_1, s, t, i, e, m)$ where $X_0, X_1 \in \mathrm{Ob}(\mathscr{C})$ represent objects and morphisms in the groupoid, $s, t : X_1 \to X_0$ source and target morphisms, $e : X_0 \to X_1$ the identity map, $i : X_1 \to X_1$ the inverse and $m : X_1 \times_{s,X_0,t} X_1 \to X_1$ composition. More formally, these are morphisms in $\mathscr{C}$ such that the following six diagrams commute:

$$
\begin{array}{ccc}
X_0 & \xrightarrow{e} & X_1 \\
{\scriptstyle e}\downarrow & \searrow^{1_{X_0}} & \downarrow{\scriptstyle t} \\
X_1 & \xrightarrow{s} & X_0
\end{array}
\qquad
\begin{array}{ccc}
X_1 & \xrightarrow{i} & X_1 \\
{\scriptstyle 1_{X_1}}\downarrow & \swarrow^{i} & \downarrow{\scriptstyle s} \\
X_1 & \xrightarrow{t} & X_0
\end{array}
\qquad
\begin{array}{ccccc}
X_1 & \xleftarrow{\mathrm{pr}_1} & X_1 \times_{s,X_0,t} X_1 & \xrightarrow{\mathrm{pr}_2} & X_1 \\
{\scriptstyle s}\downarrow & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle t} \\
X_0 & \xleftarrow{s} & X_1 & \xrightarrow{t} & X_0
\end{array}
$$

$$
\begin{array}{ccccc}
X_1 & \xrightarrow{i \times 1_{X_1}} & X_1 \times_{s,X_0,t} X_1 & \xleftarrow{1_{X_1} \times i} & X_1 \\
{\scriptstyle s}\downarrow & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle t} \\
X_0 & \xrightarrow{e} & X_1 & \xleftarrow{e} & X_0
\end{array}
$$

$$
\begin{array}{ccccc}
X_0 \times_{1_{X_0},X_0,t} X_1 & = & X_1 & = & X_1 \times_{s,X_0,1_{X_0}} X_0 \\
{\scriptstyle e \times 1_{X_1}}\downarrow & & \| & & \downarrow{\scriptstyle 1_{X_1} \times e} \\
X_1 \times_{s,X_0,t} X_1 & \xrightarrow{m} & X_1 & \xleftarrow{m} & X_1 \times_{s,X_0,t} X_1
\end{array}
$$

$$
\begin{array}{ccc}
X_1 \times_{s,X_0,t\circ\mathrm{pr}_1} (X_1 \times_{s,X_0,t} X_1) & \xrightarrow{\alpha} & (X_1 \times_{s,X_0,t} X_1) \times_{s\circ\mathrm{pr}_2,X_0,t} X_1 \\
 & & \downarrow{\scriptstyle m \times 1_{X_1}} \\
{\scriptstyle 1_{X_1} \times m}\downarrow & & X_1 \times_{s,X_0,t} X_1 \\
 & & \downarrow{\scriptstyle m} \\
X_1 \times_{s,X_0,t} X_1 & \xrightarrow{m} & X_1
\end{array}
$$

The explicit diagrams are somewhat excessive on a first pass, but useful as evidence that we may define groupoids in arbitrary categories with fibre products. The first three diagrams set up the nature of sources and targets for identities, inverses and composition respectively. The fourth represents how inverses behave in composition, the fifth doing the same for identities and the sixth insisting on associativity.

We can do this in a category of schemes and get a *groupoid in schemes*. An algebraic group is a special case of this, namely a groupoid in schemes where $X_0 = *$ and X_1 is the scheme in question. We may also define a groupoid in schemes X/G for an algebraic group G acting on a scheme X by setting $X_0 = X$, $X_1 = X \times G$, $s : (y, g) \mapsto y$, $t : (y, g) \mapsto g(y)$ and all other morphisms predictably.

How would we make this into a stack? Indeed, given a groupoid in schemes $X_\bullet = (X_0, X_1, s, t, i, e, m)$, we could convert X_0 and X_1 into stacks and all morphisms into stack morphisms. Once this has been done, we may define a presheaf in groupoids $F : \mathscr{C}^{op} \to \mathbf{Grpd}$ sending each U to $((X_0)_U, (X_1)_U, s_U, t_U, i_U, e_U, m_U)$. We denote the associated fibred category S_F as $[X_\bullet]^{ps}$ [44, pg. 100] [53, Tag 044O].

Why have we denoted this object as $[X_\bullet]^{ps}$? The superscript denotes something called a *prestack*.

Definition 2.20. [64, pg. 75] A *prestack* is a fibred category whose amalgamation functors are all fully faithful.

This is a weaker concept than a stack; the $\cdot^{fib}$ construction in fact gives an equivalence between prestacks in sets and separated presheaves [64, pg. 76].

Proposition 2.5. [44, pg. 100] If $\mathscr{C}$ is a site with subcanonical topology, then $[X_\bullet]^{ps}$ is a prestack.

Proof. Our proof is a fix to the one in [44, pg. 100]. Let $U \in \mathscr{C}$, $x, y \in (X_0)_U$. Since X_0 and X_1 are sheaves, they must also be sheaves on $\mathscr{C}/U$ by restriction. We also have the trivial sheaf $\mathscr{T}$ on $\mathscr{C}/U$ defined by $\mathscr{T}(f) = \{f\}$. This implies be the fibre product $P = \mathscr{T} \times_{X_0 \times X_0} X_1$ under maps $x \times y$ and $s \times t$ is as well. However, when we compute $P(f)$ directly for $f : T \to U$:

$$\begin{aligned} P(f : T \to U) &= \{\alpha \in (X_1)_T \mid (x \times y) \circ f = (s \times t) \circ \alpha\} \\ &= \{\alpha \in (X_1)_T \mid f^*(x) = s(\alpha), f^*(y) = t(\alpha)\} \\ &= \mathrm{Hom}_{(X_\bullet)_T}(f^*(x), f^*(y)) \\ &= \underline{\mathrm{Hom}}(x, y)(f : T \to U) \end{aligned}$$

Since P sends morphisms in $\mathscr{C}/U$ to identical ones as $\underline{\mathrm{Hom}}(x, y)$, we have that $P = \underline{\mathrm{Hom}}(x, y)$ is a sheaf. This is sufficient to show $[X_\bullet]^{ps}$ is a prestack. $\square$

The original proof in [44] used U instead of our $\mathscr{T}$, which raises problems if U has non-trivial endomorphisms.

How can we make a stack out of this prestack? Analogous to the sheafification of a presheaf, there is a process called *stackification* by which we may augment a prestack to a stack [44, pg. 100] [53, Tag 02ZP].

Proposition 2.6. [44, pg. 100] Let $F : \mathscr{S} \to \mathscr{C}$ be a prestack over a site $\mathscr{C}$ with coproducts. Then there is a morphism $\iota : \mathscr{S} \to \tilde{\mathscr{S}}$ where $\tilde{\mathscr{S}}$ is a stack, such that for every stack $\mathscr{G}$, the functor $\iota^* : \mathbf{Fun}(\tilde{\mathscr{S}}, \mathscr{G}) \to \mathbf{Fun}(\mathscr{S}, \mathscr{G})$ is an equivalence of categories.

Proof. We will explain the construction here, omitting a full proof that it works. Choose a cleavage for $\mathscr{S}$. The objects of $\tilde{\mathscr{S}}$ will be pairs $(S = \{f_i : U_i \to X\}, x)$ where S is a covering of $X \in \mathscr{C}$ and $x \in DD(S)$ in $\mathscr{S}$. A morphism $(S = \{f_i : U_i \to X\}, x) \to (S' = \{f_i' : U_i' \to X'\}, x')$ is a pair (f, f') where $f : X \to X'$ is in $\mathscr{C}$ and $f' : p^*(x) \to q^*(g^*(x'))$ is in $DD(\{g_i : U_i \times_{X'} U_i' \to X\})$, where p, q, g and this covering are defined by the commutative diagram

$$\begin{array}{ccccc} U_i \times_{X'} U_i' & \xrightarrow{\;q\;} & X \times_{X'} U_i' & \xrightarrow{\;g\;} & U_i' \\ {\scriptstyle p}\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle f_i'} \\ U_i & \xrightarrow{\;f_i\;} & X & \xrightarrow{\;f\;} & X' \end{array}$$

With this, g_i is defined as the composition $f_i \circ p$. In essence, this category consists of objects with descent data over various coverings, with morphisms defined by pulling back said objects to a common covering so they can be compared. We make $\tilde{\mathscr{S}}$ a category over $\mathscr{C}$ by sending $(\{U_i \to X\}, x)$ to X and (f, f') to f.

We now define the functor $\iota : \mathscr{S} \to \tilde{\mathscr{S}}$ by sending $x \in \mathscr{S}_U$ to $(1_U, x)$ where the latter is implicitly x's image under the amalgamation functor, and sending morphisms $\phi : x \to x'$ to $(F(\phi), \alpha)$ where α is the unique morphism satisfying the strongly cartesian condition of $(F(\phi))^*(x') \to x'$:

$$\begin{array}{ccc} x & \xrightarrow{\quad\phi\quad} & x' \\ & {\scriptstyle \exists!\,\alpha}\searrow \quad \nearrow & \\ & (F(\phi))^*(x') & \end{array}$$

□

Note that $\tilde{\mathscr{S}}$ and ι must be unique up to isomorphism [44, pg. 100]. Stackification simply adds more objects corresponding to every possible descent data to force every amalgamation functor to be essentially surjective, a perfectly natural augmentation. As noted in [44, pg. 101-102], this operation preserves being fibred in sets or groupoids up to equivalence and commutes with sheafification of separated presheaves along $\cdot^{fib}$.

Definition 2.21. The *quotient stack* of a groupoid in schemes $X_\bullet$, denoted $[X_\bullet]$, is the stackification of $[X_\bullet]^{ps}$.

This lets us define a stack corresponding to an algebraic group G acting on a scheme X, written $[X/G]$.

Understanding a stackification directly is somewhat difficult, suggesting we search for an alternative formulation specifically for the case of algebraic groups acting on schemes. Olsson in [44, ch. 26] discusses another way to interpret such quotients $[X/G]$, defining an equivalent stack whose fibre over each U consists of spans $U \leftarrow P \to X$ where P has a G-action over U, $P \to X$ is G-equivariant and there is a covering $U_i \to U$ where $P \times_U U_i \cong U_i \times G$. We call the morphism $P \to U$ a *G-torsor.* The morphisms are then G-equivariant maps $P \to P'$ respecting these spans. These will all be isomorphisms as our stack is fibred in groupoids.

While this definition may seem at a glance to contradict the intuition we approached quotient stacks with in the beginning, now consisting of what are in essence *G-principal bundles* over each U, we can still extract our original image from this interpretation. Consider the terminal scheme $*$ in our site. The fibre $[X/G]_*$ now consists of spans $* \leftarrow P \to X$ where there is a covering $U_i \to *$ which P is locally trivial on. This suggests in reality that P is just G. Hence, all such spans are really just G-equivariant maps $G \to X$, each identifying a point and its orbit. In general, each span $U \leftarrow P \to X$ now clearly identifies a generalized U-point in X and its orbit under the action of G on X, with the isomorphisms in each fibre of $[X/G]$ describing how two spans identify different points in the same orbit. It is now evident that, as said in [53, Tag 04UZ], the geometry of $[X/G]$ is the G-equivariant geometry of X.

Proposition 2.7. [44, pg. 110] For a scheme X and algebraic group G, $[X/G]$ is an Artin stack.

We won't give the full proof here, regardless of it being straightforward, as it requires algebraic spaces to prove the diagonal is representable. However, the smooth surjective map will obviously be of the form $X \to [X/G]$. The reader is directed to [44, pg. 110] for more details.

It is now rather straightforward to devise examples of quotient stacks and therefore Artin stacks. An immediate and rather satisfying one is $\mathbf{End}_n = [M_n(k)/\mathrm{GL}_n(k)]$, where the action is conjugation. This finally makes it possible to ask for geometric properties of this moduli problem, such as separatedness, quasicompactness and so on.

Analogously to moduli spaces and orbit spaces, a natural question presents itself: when do we get a quotient stack out of a moduli problem's Artin stack? Obviously we first need a group action defining the moduli problem's equivalence relation. The exact conditions that make an Artin stack a quotient stack, either globally or locally, are rather involved and completely beyond the scope of this dissertation. A major result in this direction, proven by Alper, Hall and Rydh in [2], establishes how a quasi-separated algebraic stack locally of finite type over an algebraically closed field is locally a quotient stack around points with well-behaved stabilizers. Research like this abounds; the intrigued reader is referred to [53, Tag 04UZ] and [1].

Chapter 3

Higher Category Theory

> *"If the doors of perception were cleansed every thing would appear to man as it is, Infinite. For man has closed himself up, till he sees all things thro' narrow chinks of his cavern."*
>
> – William Blake, The Marriage of Heaven and Hell

After all our celebration and excitement about the efficacy of stacks at answering our motivating question, the reader may be rightly perplexed as they read the beginning of this entirely new chapter. Unfortunately, while solving a cornucopia of problems that moduli spaces faced, stacks are still not sufficient to deal with some major challenges that we have not yet considered.

The issue becomes apparent when we consider some class of chain complexes up to *quasi-isomorphism*, namely morphisms that induce an isomorphism on cohomology [63]. Another, outside of algebraic geometry, is that of topological spaces up to homotopy equivalence. In such situations, the notion of equivalence is not immediately an isomorphism per se, so we cannot yet produce a sensible stack and declare all 'equivalent' points to be formally as such. Indeed, a morphism of stacks may be under no obligation to preserve such nuanced and implicit equivalences, being completely ignorant to the relationships we care about.

This stands as yet another step in the journey to answering our motivating question. The approach of moduli spaces works for a moduli problem $A/\sim$ when $\sim$ is represented by identities. The approach of stacks then handles the case when $\sim$ is represented by isomorphisms. Finally, we must address the most general scenario, when $\sim$ may in fact be represented by any class of morphisms.

In a rather fascinating turn of events, the answer will lie in homotopy theory. Indeed, it was discovered by Quillen [46] that homological algebraic constructions, such as our example above, can be subsumed into a more general abstract homotopy theory by *model categories*. These categories contain hand-picked classes of morphisms to be visualized as 'homotopy equivalences', which other constructions may invert in a sensible manner. However, even model categories are not sufficient for our purposes;

as noted in [58, pg. 12], there is no sensible model category of functors between two other model categories. We thus declare the theory not to be *internal*. This raises problems when we would like to think about the homotopical behavior between model categories, unlike the ease with which we describe the category **Cat**, owing to the existence of $\mathbf{Fun}(\mathscr{C}, \mathscr{D})$ for any categories $\mathscr{C}$ and $\mathscr{D}$. Furthermore, model categories will turn out to contain a great deal of superfluous information in their design, distracting from our core goal.

In our search for a more suitable theory, we will eventually realize that the solution is to track the 'higher morphisms' between morphisms, the morphisms between these, *ad infinitum*. Such morphisms should be seen as *homotopies*. Crucially, operations like composition will now only be defined 'up to homotopy', which will be key to providing the complete theory we sought. This brings us into the realm of *higher category theory*, a concept that emerged from work between the 1950's and 1980's through studies in algebraic topology, algebraic geometry and category theory [52, ch. 1]. As it did to Simpson [52], Toën and Vezzosi [57], the specific form of higher category known as a *Segal category* will draw our attention.

In the end, we will find ourselves with a stack tracking not only morphisms between its generalized points, but indeed an infinity of higher morphisms that describe our complex notion of equivalence $\sim$. Furthermore, the category of such stacks will be augmented to be a Segal category itself, containing all the higher equivalence data between stacks crucial to the theory we seek.

3.1 Localization

The progression of sections in this chapter is inspired by [58, ch. 1]. We find ourselves in a general situation where we have a pair $(\mathscr{C}, W)$ of a category and some class of morphisms representing the generalized equivalences that pique our interest. We would like, in some way, to see elements of W as the formal equivalences we intend for them to be.

A first approach is to just make all the elements of W into isomorphisms explicitly. This is the method of Gabriel and Zisman in [17], who construct a category $\mathscr{C}[W^{-1}]$ with the same objects as $\mathscr{C}$ and a functor $P_W : \mathscr{C} \to \mathscr{C}[W^{-1}]$ such that:

1. P_W sends elements of W to isomorphisms.

2. Any functor $\mathscr{C} \to \mathscr{D}$ that sends elements of W to isomorphisms factors uniquely through P_W.

This is a natural universal definition of what we will call a *localization*, or the *category of fractions* as Zisman and Gabriel name it. Such a name harks back to the classical localization of a ring, where we formally invert some submonoid of elements.

Proposition 3.1. [17, pg. 6-7] For any category $\mathscr{C}$ and collection of morphisms W, there exists a category and corresponding functor $P_W : \mathscr{C} \to \mathscr{C}[W^{-1}]$ such that for any $\mathscr{D}$ the functor $\mathbf{Fun}(P_W, \mathscr{C}) : \mathbf{Fun}(\mathscr{C}[W^{-1}], \mathscr{D}) \to \mathbf{Fun}(\mathscr{C}, \mathscr{D})$ is fully faithful and essentially surjects on all functors $\mathscr{C} \to \mathscr{D}$ sending W to isomorphisms.

Proof. Our proof is by construction, mostly from [17, pg. 6-7]. Let $T = \mathrm{Mor}(\mathscr{C}) \amalg W$, possibly a proper class. For any $f \in \mathrm{Mor}(\mathscr{C})$, let $i(f)$ be the corresponding element of T and likewise for $j(g), g \in W$. For any $f \in \mathrm{Mor}(\mathscr{C})$, let $s(f)$ be its source object and $t(f)$ its target. Then for any $f \in \mathrm{Mor}(\mathscr{C})$, define $s_T(i(f)) = s(f)$ and $t_T(i(f)) = t(f)$. Furthermore, for $g \in W$, define $s_T(j(f)) = t(f)$ and $t_T(j(f)) = s(f)$. These rules declare the value of $s_T(f)$ and $t_T(f)$ for every $f \in T$.

Now, define a category $\mathscr{C}[W^{-1}]$ whose objects are $\mathrm{Ob}(\mathscr{C})$ and morphisms are equivalence classes of finite sequences $(a_1, \dots, a_n)$ of elements in T where $t_T(a_{i-1}) = s_T(a_i)$ for all $1 < i \leq n$ under the relation $\sim$, defined such that:

1. $(\dots, i(g), i(f), \dots) \sim (\dots, i(f \circ g), \dots)$ when $f \circ g$ is defined in $\mathscr{C}$;
2. $(\dots, 1_X, \dots) \sim (\dots, \dots)$ should the left-hand side have length greater than 1;
3. $(\dots, i(g), j(g), \dots) \sim (\dots, 1_{s(g)}, \dots)$ for all $g \in W$;
4. $(\dots, j(g), i(g), \dots) \sim (\dots, 1_{t(g)}, \dots)$ for all $g \in W$.

The source and target of any such morphism is, clearly, $s_T(a_1)$ and $t_T(a_n)$. Composition is simply concatenation and identities are (1_X) for each X.

We now define a functor $P_W : \mathscr{C} \to \mathscr{C}[W^{-1}]$ defined by the obvious embedding, namely such that $X \mapsto X$ and $(f : X \to Y) \mapsto ((i(f)) : X \to Y)$. This clearly sends W to isomorphisms.

To see why it is universal, consider any functor $F : \mathscr{C} \to \mathscr{D}$ sending W to isomorphisms. We could very clearly define a functor $G : \mathscr{C}[W^{-1}] \to \mathscr{D}$ acting identically on objects, but sending a morphism (f) to $F(f)$ if $f \in \mathrm{Mor}(\mathscr{C})$ and $F(f)^{-1}$ otherwise. This is then extended by sending $(a_1, \dots, a_n)$ to $F(a_n) \circ \cdots \circ F(a_1)$. Clearly, we have $F = G \circ P_W$. Conversely, if F did not send W to isomorphisms, no such G could exist as functors always preserve isomorphisms. The uniqueness of G with regards to F is somewhat self-evident, as any other functor K where $F = K \circ P_W$ must behave identically. We will leave details to the reader. □

This construction is unfortunately not suitable for our purposes. While it commutes with finite direct limits in specific cases [17, pg. 16], many more general limits and colimits do not. Indeed, if W_I denotes all natural transformations in $\mathbf{Fun}(I, \mathscr{C})$ contained entirely in W for some category I, then $\mathbf{Fun}(I, \mathscr{C})[W_I^{-1}]$ and $\mathbf{Fun}(I, \mathscr{C}[W^{-1}])$ are usually quite different, with $\mathscr{C}[W^{-1}]$ in general not providing enough information to determine $\mathbf{Fun}(I, \mathscr{C})[W_I^{-1}]$ [61, pg. 13]. This phenomenon also makes it difficult to understand functors $(\mathscr{C}, W) \to (\mathscr{D}, V)$ that send morphisms in W to V up to natural transformations in V, since neither $\mathscr{C}[W^{-1}]$ nor $\mathscr{D}[V^{-1}]$ appear to contain enough detail, ignoring their awkward definitions.

3.2 Model Categories

In light of the failures of such a brutal localization, perhaps the pair $(\mathscr{C}, W)$ deserves a bit more direct attention. Henceforth, we will call elements of W the *weak equivalences*.

It is unclear what structure we should expect from W. For instance, W should almost certainly be closed under composition and contain all identities, as these requirements essentially describe transitivity and reflexivity of an equivalence relation. Finding an analogy to symmetry, however, is not so easy. A partial answer can be achieved in a way that encapsulates transitivity at the same time:

Definition 3.1. [48, pg. 63] A class of morphisms W is said to satisfy the *2-out-of-3 property* when, for any $f, g \in \mathrm{Mor}(\mathscr{C})$ such that $g \circ f$ is defined, if any two elements of $\{f, g, g \circ f\}$ are in W then so is the third.

This is not a complete answer, as it does not entirely enforce the existence of some 'inversion' of any $f \in W$. However, it does make clear that should a candidate inverse exist, namely some g where $g \circ f$ or $f \circ g$ is an endomorphism in W, then it must be in W along with both of these compositions.

How can we declare the existence of suitable inverses effectively? One possible answer lies in the idea of a *category of models for a homotopy theory* or *model category* for short [54], developed by Quillen in [46] to describe the similarity of homological algebra to homotopy theory in an abstract context. The goal here will be to construct 'homotopy classes' of morphisms according to W, in which context an inverse will be identical to the categorical sense. A model category is a category $\mathscr{M}$ equipped with some class of morphisms called the 'weak equivalences' W and two others, called the *fibrations* Fib and *cofibrations* Cofib.

To understand these new kinds of morphism, it is best to consider the familiar setting where $\mathscr{M} = \mathbf{Top}$ and W is the homotopy equivalences [54]. In this context, we consider the case where Fib is the class of topological (or Hurewicz) fibrations, namely those maps $p : E \to B$ such that for any $A \in \mathbf{Top}$ and commutative square

$$\begin{array}{ccc} A & \xrightarrow{f} & E \\ {\scriptstyle x\mapsto(x,0)}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ A \times [0,1] & \xrightarrow{g} & B \end{array}$$

there is a unique arrow $A \times [0,1] \to E$ sustaining the diagram's commutativity, a property we call the *homotopy lifting property* [25, pg. 375]. This is to say that any homotopy $g_t : A \to B$, given a lift $g'_0 : A \to E$ of the map g_0, can be lifted to a unique $g'_t : E \to B$. This is a natural generalization of a fibre bundle. Dually, a *cofibration* is a map $i : A \to B$ such that for any $X \in \mathbf{Top}$ and commutative square

$$\begin{array}{ccc} A & \xrightarrow{f} & E^{[0,1]} \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle x\mapsto x(0)} \\ B & \xrightarrow{g} & E \end{array}$$

where $E^{[0,1]}$ is the space of continuous maps $[0,1] \to E$, there exists a unique arrow $B \to E^{[0,1]}$ maintaining commutativity, which we call the *homotopy extension property* [25, pg. 460]. A *closed cofibration* is then one whose image is closed. The class of all such maps will be titled Cofib.

We will generalize the properties above to more abstract 'lifting properties', predictably titled the *right lifting property* (RLP) and *left lifting property* (LLP) with respect to some class of morphisms. A morphism f has the LLP with respect to g and g the RLP with respect to f if a dotted arrow exists in any commutative diagram, called a *lifting problem*, of the form

$$\begin{array}{ccc} A & \longrightarrow & B \\ {\scriptstyle f}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle g} \\ C & \longrightarrow & D \end{array}$$

This relationship is written as $f \boxslash g$ [48, pg. 37].

We can then say that a class of morphisms A has the right (left) lifting property with respect to another class B, written $A \boxslash B$, exactly when $f \boxslash g$ for every $f \in A$ and $g \in B$ [48, pg. 37].

Definition 3.2. [48, pg. 37] Given a class of morphisms A, we define its *right complement* $A^{\boxslash}$ and *left complement* ${}^{\boxslash}A$ by

$$A^{\boxslash} = \{f \mid \forall a \in A \bullet a \boxslash f\}$$
$${}^{\boxslash}A = \{f \mid \forall a \in A \bullet f \boxslash a\}$$

It is by definition that $A \boxslash A^{\boxslash}$ and ${}^{\boxslash}A \boxslash A$ for any class of morphisms A; indeed, these are the maximal classes of morphisms for which this is so.

Proposition 3.2. [54, pg. 2] Cofib $= {}^{\boxslash}(\text{Fib} \cap W)$ and $\text{Cofib}^{\boxslash} = (\text{Fib} \cap W)$ in our above example. Furthermore, Fib $= (\text{Cofib} \cap W)^{\boxslash}$ and ${}^{\boxslash}\text{Fib} = (\text{Cofib} \cap W)$.

We refer the reader to [54] for a full proof, though this itself refers to proofs elsewhere.

Another more striking property of Fib and Cofib is the following:

Proposition 3.3. [54, pg. 2] Any morphism $f : X \to Y$ in **Top** is of the form $f = p \circ i = p' \circ i'$, where $p \in$ Fib, $i \in (\text{Cofib} \cap W)$, $p' \in (\text{Fib} \cap W)$ and $i' \in$ Cofib.

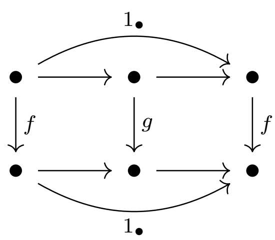
Again, the proof may be found in [54], requiring some algebraic topology that lies beyond the scope of this dissertation. We generalize the above two propositions to a definition:

Definition 3.3. [48, pg. 39] A pair of morphism classes (A, B) are said to be a *weak factorization system* if $A = {}^{\boxslash}B$, $B = A^{\boxslash}$ and every morphism f in $\mathscr{C}$ is of the form $f = b \circ a$, where $b \in B$ and $a \in A$.

It is clear that both $(\text{Cofib}, \text{Fib} \cap W)$ and $(\text{Cofib} \cap W, \text{Fib})$ form weak factorization systems.

In any weak factorization system, both fibrations and cofibrations contain all isomorphisms and are closed under composition, while cofibrations are closed under coproducts and pushouts and fibrations under the dual cases [48, pg. 37]. Clearly isomorphisms satisfy lifting for everything and lifts may be composed, so this is all self-evident or simple enough for us not to explicitly prove here. A perhaps more interesting property is the following, a generalization of retractions of maps in **Top**:

Definition 3.4. [48, pg. 34] A morphism f is called a *retract* of g if there is a commutative diagram of the form



Both classes in a weak factorization system are closed under retract [48, pg. 37]; the proof is not hard, though we will not give it here.

Before we try to explain ourselves a little better, we might as well form the central definition at this point:

Definition 3.5. [48, pg. 126] A *model category* is a category $\mathscr{M}$ with three classes of morphisms, W, Fib and Cofib, such that $\mathscr{M}$ has all small limits and colimits, W satisfies the 2-out-of-3 property and both $(\text{Cofib}, \text{Fib} \cap W)$ and $(\text{Cofib} \cap W, \text{Fib})$ are weak factorization systems.

In a general model category, we will call elements of $\text{Fib} \cap W$ *acyclic fibrations* and $\text{Cofib} \cap W$ *acyclic cofibrations* [15, pg. 12].

3.2.1 Homotopies in Model Categories

Our goal is to try and generalize homotopy classes of maps from the topological case we have considered to far to more general scenarios. While somewhat odd, the addition of fibrations and cofibrations to the picture will turn out to be a wise investment, an invisible structural foundation ensuring our abstract notion of homotopy will behave as we expect.

For any model category $\mathscr{M}$, the existence of small limits and colimits implies the existence of initial and terminal objects $\emptyset$ and $*$ respectively. With this in mind, we may define a *fibrant object* A as one where the unique map $A \to *$ is a fibration and a *cofibrant object* as one where $\emptyset \to A$ is a cofibration [15, pg. 13]. An object that has both properties will be called *fibrant-cofibrant.*

Note that any object $X \in \mathscr{M}$ is weakly equivalent to a fibrant object, as we may factorize the map $X \to *$ into an acyclic cofibration $X \to FX$ and fibration $FX \to *$. Similarly, any object is weakly equivalent to a cofibrant object $CX \to X$. Notice that the fibrant object we get from doing this to a cofibrant one remains cofibrant, as cofibrations are closed under composition. Hence, any object has at least one *fibrant-cofibrant replacement* FCX it is weakly equivalent to. This will turn out to be rather handy later.

In the case of **Top**, a fibrant object would be any A such that for any span $B \times [0,1] \leftarrow B \to A$, there is a lift $B \times [0,1] \to A$ commuting with the span. Indeed, this could in general be done by remaining constant on $[0,1]$, so all objects are fibrant. A cofibrant object would then be any C such that for any cospan $B^{[0,1]} \to B \leftarrow C$, there is a morphism $C \to B^{[0,1]}$ that the cospan factors through. This could also be achieved by constant maps, so every object is also cofibrant.

This doesn't seem particularly fascinating - the fibrant-cofibrant objects are just everything. We should not be surprised however, as this simply indicates that whatever abstract notion of homotopy we engineer that depends on fibrant or cofibrant objects will work completely seamlessly in **Top**. The picture becomes somewhat more interesting in Quillen's original model category structure on **Top** [46], where the weak equivalences are *weak homotopy equivalences*, namely those that induce isomorphisms on all homotopy groups. Both singular homology and cohomology are preserved by homotopy groups [25, pg. 356], so in some sense this equivalence represents the best approximation of a space that algebraic topology cares for.

What are the appropriate fibrations and cofibrations? Quillen [46] declares the former to be the *Serre fibrations*, namely the class of morphisms with the RLP with respect to the inclusions $(1_{D^n}, 0) : D^n \hookrightarrow (D^n \times [0, 1])$ for all n [25, pg. 376]. The cofibrations are then the left complement of these, which turn out to be exactly the retracts of maps $X \to Y$ where Y is obtained from X by attaching cells [15, pg. 13]. It does indeed turn out that this is a model category [46]. When we eventually take this category 'up to homotopy', we will wind up with the homotopy category of CW complexes. Indeed, every space is weakly homotopy equivalent to a CW complex [25, pg. 352] and Whitehead's theorem establishes that the weak homotopy equivalences between CW complexes are exactly the homotopy equivalences therein [25, pg. 346].

This case was interesting to consider, as the retracts of CW complexes here are in fact exactly the fibrant-cofibrant objects: the fibrant objects are again everything clearly, but if $\emptyset \to A$ is to be a cofibration then A must be a retract of a CW-complex. As with all model categories, every object in **Top** is weakly equivalent to some fibrant-cofibrant object, the restriction to which turns our weak equivalence into a familiar homotopy theory.

It is precisely the seeming coincidence that fibrant and cofibrant objects render weak equivalences so similar to classical homotopy theory which model categories seek to generalize. There are two natural notions of a homotopy between maps, titled *left homotopies* and *right homotopies* respectively; we will see how fibrant and cofibrant replacements help make them act as we should expect.

3.2.1.1 Left Homotopy

Our first notion of homotopy stems from 'cylinder objects', taking inspiration from the space $A \times [0, 1]$ for any $A \in$ **Top**. Recall that two maps $f, g : A \to B$ could be called homotopic if the map $(f + g) : A \amalg A \to B$ extends to a map $F : A \times [0, 1] \to B$ such that $F(-, 0) = f$ and $F(-, 1) = g$.

Definition 3.6. [15, pg. 18] A *cylinder object* of an object A is an object $A \wedge I$ in the diagram

$$A \amalg A \xrightarrow{i} A \wedge I \xrightarrow{w} A$$

where w is a weak equivalence, factoring the fold map $(1_A + 1_A) : A \amalg A \to A$. This is called a *good cylinder object* if i is a cofibration and a *very good cylinder object* if w is a fibration. For any such object, write $i_0, i_1 : A \to A \wedge I$ for the compositions $i \circ \text{inc}_0$ and $i \circ \text{inc}_1$, where inc_i is the i^{th} inclusion of A into the coproduct.

The crucial distinction of the cylinder object $A \wedge I$ from the coproduct $A \amalg A$ is its weak equivalence to A. Note also that every object has a very good cylinder object, though this may not be unique, nor a product of A with any sensible 'interval' object.

Lemma 3.1. [15, pg. 18] If A is cofibrant and $A \wedge I$ a good cylinder object for A, then the maps i_0, i_1 are acyclic cofibrations.

Proof. We check this for i_0, as i_1 is proven identically. Note that the identity 1_A factors as $A \xrightarrow{\text{inc}_0} A \amalg A \to A$, which means it factors as $A \xrightarrow{i_0} A \wedge I \xrightarrow{w} A$ for some weak equivalence w. This implies by the 2-out-of-3 property that i_0 is a weak equivalence. From [48, pg. 34], one of the properties cofibrations are stable under is cobase change. Hence, the pushout diagram

$$\begin{array}{ccc} \emptyset & \longrightarrow & A \\ \downarrow{\scriptstyle \in \text{Cofib}} & & \downarrow{\scriptstyle \text{inc}_0} \\ A & \xrightarrow{\text{inc}_1} & A \amalg A \end{array}$$

implies that inc_0 is a cofibration. This means i_0, the composition of two cofibrations, is itself a cofibration. □

This leads us to our new notion of homotopy:

Definition 3.7. [15, pg. 19] Two maps $f, g : A \to B$ are *left homotopic*, written $f \sim_l g$, if there exists a cylinder object $A \wedge I$ for A and some $H : A \wedge I \to B$ such that $H \circ (i_0 + i_1) = (f + g)$. We now call H a *left homotopy* of f and g, which is then called *good* or *very good* if the cylinder object is as such.

Note that any left homotopy can be replaced with a good left homotopy, by using the weak factorization property on i and recalling weak equivalences to be closed under composition. In fact, this can be taken further:

Lemma 3.2. [15, pg. 19] If B is fibrant, any left homotopy of $f, g : A \to B$ can be replaced with a very good left homotopy.

Proof. Choose a good left homotopy $H : A \wedge I \to B$ from f to g. We can factor the weak equivalence $A \wedge I \to A$ into an acyclic fibration and cofibration $A \wedge I \to A \wedge I' \to A$. By the 2-out-of-3 property, the cofibration is also acyclic. Composing this with the map $i : A \amalg A \to A \wedge I$ gives us a very good cylinder object $A \wedge I'$. Now, consider the commutative diagram

$$\begin{array}{ccc} A \wedge I & \xrightarrow{H} & B \\ \downarrow & & \downarrow \\ A \wedge I' & \longrightarrow & * \end{array}$$

As B is fibrant and the map $A \wedge I \to A \wedge I'$ is an acyclic cofibration, there is a lift $H' : A \wedge I' \to B$ commuting with the square. Clearly, $H' \circ (i_0' + i_1') = (f + g)$ by construction. □

We should expect $\sim_l$ to be some kind of equivalence relation, should our category be locally small. Indeed, for reflexivity, A is in fact a cylinder object for A, meaning $f : A \to B$ is a left homotopy between f and f. Furthermore, the map $s = (\mathrm{in}_1 + \mathrm{in}_0) : A \amalg A \to A \amalg A$ swapping components gives us a new cylinder object via $i \circ s : A \amalg A \to A \wedge I$. We can then use the identity $(g + f) = (f + g) \circ s$ to convert a left homotopy $f \sim_l g$ to $g \sim_l f$.

Transitivity is slightly more complex. The proof becomes possible if we assume that our domain is in fact cofibrant:

Proposition 3.4. [15, pg. 19-20] If A is cofibrant, then $\sim_l$ is an equivalence relation on $\mathrm{Hom}(A, B)$.

Proof. We only need to prove transitivity. Let $f \sim_l g$ and $g \sim_l h$. Let $H : A \wedge I \to B$ be a good left homotopy from f to g and $H' : A \wedge I' \to B$ a good left homotopy from g to h. Consider the span $i_1 : A \wedge I \leftarrow A \to A \wedge I' : i_0'$, whose pushout will be titled $A \wedge I''$. The span contains only acyclic cofibrations as A is cofibrant. As cofibrations are stable under cobase change, by the universality of pushouts $A \wedge I''$ is a cylinder object for A. This universality can be applied again to H and H' to give a map $H'' : A \wedge I'' \to B$ that is a left homotopy from f to h. □

We write $\pi_l(A, B)$ for the quotient of $\mathrm{Hom}(A, B)$ by the equivalence relation $\sim_l$ generates. It is indeed important that we say 'generates' here, since in general A may not be cofibrant [15, pg. 20].

With this new construction available to us, we can now prove some interesting results with regards to composition up to left homotopy:

Proposition 3.5. [15, pg. 20] If A is cofibrant and $p : B \to C$ an acyclic fibration, there is an induced bijection $p_* : \pi_l(A, B) \to \pi_l(A, C)$, such that $[f] \mapsto [p \circ f]$.

Proof. Note that for any two maps $f, g : A \to B$ such that $f \sim_l g$ by H, we have $p \circ f \sim_l p \circ g$ by $p \circ H$, which implies that p_* is well defined. We now wish to show it is surjective and injective. For the former, let $[f] \in \pi_l(A, C)$. Then we have a diagram

$$\begin{array}{ccc} \emptyset & \longrightarrow & B \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ A & \xrightarrow{f} & C \end{array}$$

Since A is cofibrant and p is an acyclic fibration, we have a lift $g : A \to B$. This means $p_*([g]) = [p \circ g] = [f]$ by commutativity, proving surjectivity.

To prove p_* is injective, consider some $f, g : A \to B$ such that $p \circ f \sim_l p \circ g$. Choose a good left homotopy $H : A \wedge I \to C$ representing this. We then have a lift in the diagram

$$\begin{array}{ccc} A \amalg A & \xrightarrow{(f+g)} & B \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ A \wedge I & \longrightarrow & C \end{array}$$

which, by commutativity, gives us the left homotopy from f to g we sought. □

Proposition 3.6. [15, pg. 21] Suppose for some fibrant B that maps $f, g : A \to B$ are left homotopic. Then for any $h : C \to A$, we have $f \circ h \sim_l g \circ h$.

Proof. The fact that B is fibrant grants us access to a very good left homotopy $H : A \wedge I \to B$ from f to g. Next, choose a good cylinder object $C \wedge I$ for C, with the cofibrant map being $j : C \amalg C \to C \wedge I$.

We can now inspect the lift of the diagram

$$\begin{array}{ccc} C \amalg C & \xrightarrow{i \circ (h \amalg h)} & A \wedge I \\ {\scriptstyle j}\big\downarrow & & \big\downarrow{\scriptstyle w} \\ C \wedge I & \xrightarrow{h \circ w'} & A \end{array}$$

where w and w' are the weak equivalences of $A \wedge I$ and $C \wedge I$, with the former a fibration. This gives us a map $k : C \wedge I \to A \wedge I$. It is not hard to verify that $H \circ k$ is the desired homotopy. □

Proposition 3.7. [15, pg. 21] If C is fibrant, composition in the model category induces a map

$$\pi_l(A, B) \times \pi_l(B, C) \to \pi_l(A, C)$$

defined as $([f], [g]) \mapsto [g \circ f]$.

Proof. Crucially, as noted in [15, pg. 21], we do not have that B and A are cofibrant, so if $[f] = [g]$ in either $\pi_l(A, B)$ or $\pi_l(B, C)$ then it is not implied that $f \sim_l g$. Regardless, it will suffice to show that if $f, g : A \to B$ are such that $f \sim_l g$ and $m, n : B \to C$ are such that $m \sim_l n$, we have $[m \circ f] = [n \circ g]$ in $\pi_l(A, C)$.

To do so, we may prove that $(m \circ f) \sim_l (m \circ g) \sim_l (n \circ g)$. The first homotopy is obtained by composing the homotopy between f and g with m, while the second follows from Proposition 3.6. □

Why are fibrant and cofibrant objects continuously coming to our rescue? The answer is, of course, due to the lifting problems they let us solve and the innumerable closure properties that the fibrations and cofibrations satisfy. The image of a fibrant or cofibrant replacement should be racing through the reader's mind at this point.

3.2.1.2 Right Homotopy

Dual to our above discussion we next consider 'path objects', deriving from the object $B^{[0,1]}$ in **Top**. Two maps $f, g : A \to B$ may be declared homotopic if the product map $(f, g) : A \to B \times B$ lifts to a map $F : A \to B^{[0,1]}$ such that $(f, g)(x) = (F(x)(0), F(x)(1))$ for all $x \in A$.

Definition 3.8. [15, pg. 22] A *path object* for A is an object A^I together with a diagram

$$A \xrightarrow{w} A^I \xrightarrow{p} A \times A$$

where w is a weak equivalence, factoring the map $(1_A, 1_A) : A \to A \times A$. We call this a *good path object* if p is a fibration and a *very good path object* if w is an acyclic cofibration. Define maps $p_0, p_1 : A^I \to A$ by $p_0 = (\mathrm{pr}_0 \circ p)$ and $p_1 = (\mathrm{pr}_1 \circ p)$.

Similarly to cylinder objects, such an object always exists, though will in general not be unique.

Definition 3.9. [15, pg. 22] Two maps $f, g : A \to B$ are said to be *right homotopic*, denoted $f \sim_r g$, if there exists a path object B^I for B and some $H : A \to B^I$ such that $(p_0, p_1) \circ H = (f, g)$. The map H is called a *right homotopy* of f and g, which is then declared to be *good* or *very good* if the path object is so.

This definition is completely dual to that of left homotopy, which leads to the both fortunate and dull implication that all our results for left homotopy may be dualized to ones here. We will list all the appropriate results from [15, pg. 22] below.

Proposition 3.8. If B is fibrant and B^I is a good path object, then p_0 and p_1 are acyclic fibrations.

Proposition 3.9. For any two maps $f, g : A \to B$ such that $f \sim_r g$, there is a good right homotopy from f to g. If in addition A is cofibrant, this can be made a very good right homotopy.

Proposition 3.10. $\sim_r$ is a symmetric and reflexive relation on any $\mathrm{Hom}(A, B)$. If B is fibrant, it is an equivalence relation.

We now define $\pi_r(A, B)$ to be the quotient $\mathrm{Hom}(A, B)/\sim'_r$, where $\sim'_r$ is the equivalence relation generated by $\sim_r$.

Proposition 3.11. If C is fibrant and $i : A \to B$ is an acyclic cofibration, then there is an induced bijection $i^* : \pi_r(B, C) \to \pi_r(A, C)$ by precomposition.

Proposition 3.12. If A is cofibrant and $f, g : A \to B$ are right homotopic maps, then for any $h : B \to C$ we have $h \circ f \sim_r h \circ g$.

Proposition 3.13. If A is cofibrant then composition in the model category induces a map $\pi_r(A, B) \times \pi_r(B, C) \to \pi_r(A, C)$ for any objects B, C.

3.2.1.3 Relating Left and Right Homotopies

Having seen two distinct notions of homotopy that really represent the same concept in **Top**, we may be curious as to when they agree in general. Indeed, as we have seen several times already, such an appeal to intuition becomes reality in the context of fibrant and cofibrant objects:

Theorem 3.1. [15, pg. 23] Let $f, g : A \to B$ be maps.

1. If A is cofibrant and $f \sim_l g$, then $f \sim_r g$.
2. If B is fibrant and $f \sim_r g$, then $f \sim_l g$.

Proof. These two statements are dual, so we will only prove the first. Take a good left homotopy $H : A \wedge I \to B$ from f to g. Choose a good path object B^I for B. Let $j : A \wedge I \to A$ and $q : B \to B^I$ be the weak equivalences in each respective object's definition.

Consider the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{q \circ f} & B^I \\ {\scriptstyle i_0}\downarrow & & \downarrow{\scriptstyle (p_0, p_1)} \\ A \wedge I & \xrightarrow{(f \circ j, H)} & B \times B \end{array}$$

Since p_0 and p_1 are both fibrations, they have the RLP with regards to all acyclic cofibrations. Any such lifting problem for (p_0, p_1) can be broken down to one for each such map and then combined by universality to show (p_0, p_1) is a fibration too. Hence, as i_0 is an acyclic cofibration, there is a lift $K : A \wedge I \to B^I$ in this diagram.

Note that $(p_0, p_1) \circ (K \circ i_1) = (f \circ j, H) \circ i_1 = (f \circ j \circ i_1, H \circ i_1)$ by the above diagram. This is then equal to (f, g) by definition, so $K \circ i_1$ was the right homotopy we sought. $\square$

We will refer to the equivalence relation generated by both left and right homotopy as *homotopy*, or $\sim$. The quotient of $\mathrm{Hom}(A, B)$ by this relation will be written as $\pi(A, B)$. If A is cofibrant and B is fibrant, we can see that $\sim$ is both $\sim_l$ and $\sim_r$ simultaneously.

It is about time we took a step back and considered how this relates to our original problem of finding inverses for morphisms in W. Finally, the answer begins to creep up on us:

Lemma 3.3. [15, pg. 23] Suppose $f : A \to B$ is a morphism, where A and B are both fibrant-cofibrant. Then f is a weak equivalence if and only if it has a *homotopy inverse*, namely some $g : B \to A$ such that $g \circ f \sim 1_A$ and $f \circ g \sim 1_B$.

Proof. Suppose first that f is a weak equivalence. We can of course factor $f = p \circ i$, where $i : A \to C$ is an acyclic cofibration and $p : C \to B$ a fibration. p is also acyclic by the 2-out-of-3 property. Consider the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{1_A} & A \\ {\scriptstyle i}\downarrow & & \downarrow \\ C & \longrightarrow & * \end{array}$$

Since A is fibrant, there is a lift $r : C \to A$ such that $r \circ i = 1_A$, ie. a left inverse. By Propsition 3.11, we have a bijection $i^* : \pi_r(C, B) \to \pi_r(A, B)$ induced by precomposition. Note that $i^*([i \circ r]) = [i \circ r \circ i] = [i] = i^*([1_A])$, so $i \circ r \sim_r 1_A$. Hence, r is a two-sided homotopy inverse for i. Dualizing this argument gives us a two-sided homotopy inverse s for p, resulting in the two-sided homotopy inverse $r \circ s$ for $p \circ i = f$.

Conversely, suppose f has a two-sided homotopy inverse, say $g : B \to A$. Note that if we factorize $f = p \circ i$ into a fibration $p : C \to B$ and acyclic cofibration $i : A \to C$ respectively, it will suffice by the 2-out-of-3 property to prove p is a weak equivalence. Fortunately, C is cofibrant since $(\emptyset \to C) = i \circ (\emptyset \to A)$ and fibrant since $(C \to *) = (B \to *) \circ p$.

Let $H : B \wedge I \to B$ form a left homotopy from $f \circ g$ to 1_B. Let $H' : B \wedge I \to C$ be the lift in the diagram

$$\begin{array}{ccc} B & \xrightarrow{i\circ g} & C \\ {\scriptstyle i_0}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ B \wedge I & \xrightarrow{H} & B \end{array}$$

Let $s = H' \circ i_1$, so $p \circ s = p \circ H' \circ i_1 = H \circ i_1 = 1_B$ by the diagram above. Since i is a weak equivalence, we have just shown that it must have a homotopy inverse, which we call $r : C \to A$. As $p \circ i = f$, we have via right composition with r that $p \sim f \circ r$ by Proposition 3.6. Note also that H' is a left homotopy in the above diagram witnessing $s \sim i \circ g$. Therefore, by the composition maps in Propositions 3.7 and 3.13, we have

$$(s \circ p) \sim (i \circ g \circ p) \sim (i \circ g \circ f \circ r) \sim (i \circ r) \sim 1_C$$

Note that if $K : C \wedge I \to C$ is the left homotopy between $s \circ p$ and 1_C, the maps i_0, i_1 are of course weak equivalences. Hence, since $1_C = K \circ i_1$, we have that K is a weak equivalence, so $s \circ p = K \circ i_0$ is one too.

We will now switch to the proof from [40, prop. 4.1] to show that, since p is a retract of $s \circ p$ by the diagram

$$\begin{array}{ccccc} & & \overset{1_C}{\curvearrowright} & & \\ C & \xrightarrow{1_C} & C & \xrightarrow{1_C} & C \\ {\scriptstyle p}\big\downarrow & & {\scriptstyle s\circ p}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ B & \xrightarrow{s} & C & \xrightarrow{p} & B \\ & & \underset{1_B}{\smile} & & \end{array}$$

then p is a weak equivalence too. In general, weak equivalences are closed under retract by this proposition, though we may prove it more easily here because p is a fibration.

Factor $s \circ p = m \circ n$, where m is an acyclic fibration and n a cofibration, which is also acyclic by 2-out-of-3. Then consider the commutative diagram

$$\begin{array}{ccccc} & & \overset{1_C}{\curvearrowright} & & \\ C & \xrightarrow{1_C} & C & \xrightarrow{1_C} & C \\ {\scriptstyle 1_C}\big\downarrow & & {\scriptstyle n}\big\downarrow & & \big\downarrow{\scriptstyle 1_C} \\ C & \xrightarrow{n} & X & \xrightarrow{v} & C \\ {\scriptstyle p}\big\downarrow & & {\scriptstyle m}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ B & \xrightarrow{s} & C & \xrightarrow{p} & B \\ & & \underset{1_B}{\smile} & & \end{array}$$

where v is the lift of the commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{1_C} & C \\ \big\downarrow{\scriptstyle n} & & \big\downarrow{\scriptstyle p} \\ X & \xrightarrow{p \circ m} & B \end{array}$$

Because $v \circ n = 1_C$, p is a retract of an acyclic fibration. As we know from weak factorization systems, acyclic fibrations are closed under retracts, so p is a weak equivalence. □

3.2.2 The Homotopy Category of a Model Category

With a reasonable notion of homotopy in hand, we would like to somehow consider our model category 'up to homotopy', a category with homotopy classes of maps for morphisms. Indeed, as we have observed, doing this should convert exactly the weak equivalences into isomorphisms, letting us manipulate weak equivalences with traditional category theory.

However, we find ourselves at an irritating impasse: all of our results depend in some way or other on the involved objects being fibrant or cofibrant. In order to leverage this machinery, we will need to somehow restrict ourselves to these cases.

In a general category, consider two isomorphisms, $w : A \to A'$ and $v : B \to B'$. For any morphism $f : A \to B$, there is of course a unique arrow $\tilde{f}$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle w}\big\downarrow & & \big\downarrow{\scriptstyle v} \\ A' & \xrightarrow{\tilde{f}} & B' \end{array}$$

commutes. Hence, if we so wished, we could consider $\mathrm{Hom}(A, B)$ and $\mathrm{Hom}(A', B')$ interchangeably. Now imagine that w and v were instead weak equivalences and A' and B' were some fibrant-cofibrant replacements of A and B. We would similarly like to see an interchangeability of $\pi(A, B)$ and $\pi(A', B')$, letting us effectively replace the former with the latter in all cases. Fortunately, it is exactly this that we will be able to do.

Henceforth, for any X, let $X \overset{i_X}{\to} RX$ be some fibrant object weakly equivalent to X via an acyclic cofibration i_X and $QX \overset{p_X}{\to} X$ some cofibrant object via an acyclic fibration p_X. If X is already fibrant or cofibrant, let these be equal to X. As previously mentioned, such replacements always exist, though may not be unique.

Lemma 3.4. [15, pg. 25] Given a morphism $f : X \to Y$ there exists a morphism $\tilde{f} : QX \to QY$ such that the diagram

$$\begin{array}{ccc} QX & \xrightarrow{\tilde{f}} & QY \\ {\scriptstyle p_X}\downarrow & & \downarrow{\scriptstyle p_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

commutes. The map $\tilde{f}$ is unique up to left homotopy and up to right homotopy and is a weak equivalence exactly when f is.

Proof. We construct $\tilde{f}$ by taking a lift of the diagram

$$\begin{array}{ccc} \emptyset & \longrightarrow & QY \\ \downarrow & & \downarrow{\scriptstyle p_Y} \\ QX & \xrightarrow{f \circ p_X} & Y \end{array}$$

That this is unique up to left homotopy follows from Proposition 3.5. For right homotopy, because QX is cofibrant, Theorem 3.1 implies the result. The weak equivalence condition follows from 2-out-of-3. □

We get a dual statement about fibrant replacements, so omit the dualized proof.

Lemma 3.5. [15, pg. 25] Given some morphism $f : X \to Y$ there is a morphism $\overline{f} : RX \to RY$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle i_X}\downarrow & & \downarrow{\scriptstyle i_Y} \\ RX & \xrightarrow{\overline{f}} & RY \end{array}$$

commutes. The map $\overline{f}$ is unique up to left homotopy and up to right homotopy and is a weak equivalence exactly when f is.

With both of these in mind, for any objects A, B, we may always replace $\pi(A, B)$ with $\pi(QA, QB)$ or $\pi(RA, RB)$ whenever we wish. To make homotopy behave as naturally as possible, we will replace $\pi(A, B)$ with $\pi(RQA, RQB)$. We get a natural map $\mathrm{Hom}(A, B) \to \pi(RQA, RQB)$ by sending $f \mapsto [\overline{\tilde{f}}]$.

Note that, for any A, B, C, there is a map induced by composition

$$\pi(RQA, RQB) \times \pi(RQB, RQC) \to \pi(RQA, RQC)$$

by Proposition 3.7 or 3.13, commuting with the composition map $\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$ and the map we defined. We now find ourselves with sufficiently sturdy foundations for a *homotopy category*.

Definition 3.10. [15, pg. 26] Let $\mathscr{M}$ be a model category. Define the *homotopy category* $\mathbf{Ho}(\mathscr{M})$ as the category whose objects are $\mathrm{Ob}(\mathscr{M})$ and with morphisms

$$\mathrm{Hom}_{\mathbf{Ho}(\mathscr{M})}(A, B) := \pi(RQA, RQB)$$

and composition is defined by the aforementioned map. Define $\gamma : \mathscr{M} \to \mathbf{Ho}(\mathscr{M})$ to be the identity on objects and $\gamma(f) = [\bar{\tilde{f}}]$ for a morphism f.

To ensure we can discuss a particular $\mathbf{Ho}(\mathscr{M})$, we are implicitly assuming the existence of a mapping $X \mapsto RQX$ for all $X \in \mathscr{M}$. No axiom of choice can really save us here if $\mathscr{M}$ is large. We will ignore such technicalities here, as not only will most cases have a clear mapping $X \mapsto RQX$, but we will in fact see later how to construct a category isomorphic to $\mathbf{Ho}(\mathscr{M})$ always, along with a functor γ.

A point raised in [15, pg. 27] is that swapping RQA with QRA would indeed not change this definition at all. Verifying this formally is done by finding a bijection $\pi(RQA, RQB) \to \pi(QRA, QRB)$ that agrees with the induced composition maps in both cases. The bijection would take the form $[\bar{\tilde{f}}] \mapsto [\tilde{\bar{f}}]$ for any f, which we do not verify here.

If we defined $\mathbf{Ho}(\mathscr{M})$ correctly, we should expect exactly the weak equivalences to be sent by γ to ismorphisms. The reader may now sigh in relief as we verify this motivating property of model categories:

Proposition 3.14. [15, pg. 27] For any morphism f, $\gamma(f)$ is an isomorphim if and only if f is a weak equivalence. The morphisms of $\mathbf{Ho}(\mathscr{M})$ are generated by composition of the images under γ of morphisms in $\mathscr{M}$ and inverses of images under γ of weak equivalences in $\mathscr{M}$.

Proof. If $f : A \to B$ is a weak equivalence, then $\bar{\tilde{f}}$ is a weak equivalence, as we have proven. This means it has a homotopy inverse $g : B \to A$, which in $\mathbf{Ho}(\mathscr{M})$ will be a formal inverse as $[g] \circ [\bar{\tilde{f}}] = [1_A]$ and $[\bar{\tilde{f}}] \circ [g] = [1_B]$. Conversely, $\gamma(f)$ having a formal inverse is exactly the condition to have a homotopy inverse, making f a weak equivalence.

Note for any object A in $\mathscr{M}$, the map $\gamma(i_{QA}) \circ \gamma(p_A)^{-1}$ is an isomorphism in $\mathbf{Ho}(\mathscr{M})$ from A to RQA. Moreover, given any A, B in $\mathscr{M}$, γ will of course induce a surjection from $\mathrm{Hom}(RQA, RQB)$ to $\mathrm{Hom}_{\mathbf{Ho}(\mathscr{M})}(RQA, RQB)$. Hence, any $f : A \to B$ in $\mathbf{Ho}(\mathscr{M})$ is of the form

$$f = \gamma(p_B) \circ \gamma(i_{QB})^{-1} \circ \gamma(f') \circ \gamma(i_{QA}) \circ \gamma(p_A)^{-1}$$

for some $f' : RQA \to RQB$ in $\mathscr{M}$. $\square$

This property is strangely reminiscent of the localizations from Gabriel and Zisman we saw earlier. As a matter of fact, that turns out to be exactly what we have constructed. We will first need a helpful lemma:

Lemma 3.6. [15, pg. 27-28] For any model category $\mathscr{M}$ and functor $F : \mathscr{M} \to \mathscr{C}$ taking weak equivalences in $\mathscr{M}$ to isomorphisms in $\mathscr{C}$, $f \sim_l g$ or $f \sim_r g$ implies $F(f) = F(g)$ for all $f, g : A \to B$ in $\mathscr{M}$.

Proof. We prove for the case $f \sim_l g$, as $f \sim_r g$ is dual. Let $H : A \wedge I \to B$ be a left homotopy from f to g, with the weak equivalence $w : A \wedge I \to A$ defining our cylinder object. As $w \circ i_0 = w \circ i_1$, we have $F(w) \circ F(i_0) = F(w) \circ F(i_1)$, which means $F(i_0) = F(i_1)$ as $F(w)$ is an isomorphism. Hence, $F(f) = F(H) \circ F(i_0) = F(H) \circ F(i_1) = F(g)$. □

Proposition 3.15. [15, pg. 29] For any model category $\mathscr{M}$, $\mathbf{Ho}(\mathscr{M}) \cong \mathscr{M}[W^{-1}]$ by an isomorphism that makes γ and P_W commute.

Proof. All we must do is show that γ is a localization itself. If so, universality of P_W will demand the existence of a unique functor F such that $\gamma = F \circ P_W$, which when combined with the unique functor G such that $P_W = G \circ \gamma$ from γ being a localization will imply F and G are mutually inverse.

We already know that $\gamma(f)$ is an isomorphism for all $f \in W$. Now we must prove universality. Consider any $F : \mathscr{C} \to \mathscr{D}$ sending W to isomorphisms. We must show there to be a unique $G : \mathbf{Ho}(\mathscr{M}) \to \mathscr{D}$ such that $F = G \circ \gamma$. As $\mathbf{Ho}(\mathscr{M})$ has identical objects to $\mathscr{M}$, we are required to set $G(X) = F(X)$ for all $X \in \mathrm{Ob}(\mathscr{M})$. We now only need to decide where morphisms go.

Let $f : A \to B$ be a morphism in $\mathbf{Ho}(\mathscr{M})$, represented by some $f' : RQA \to RQB$ up to homotopy. We know $F(f')$ depends entirely on the homotopy class of f' by our lemma and thus only on f. Define $G(f)$ as

$$G(f) = F(p_B) \circ F(i_{QB})^{-1} \circ F(f') \circ F(i_{QA}) \circ F(p_A)^{-1}$$

where the inverses are of course defined since $i_{QB}, p_A \in W$. It is somewhat immediate that $G(f \circ g) = G(f) \circ G(g)$ by this definition. Furthermore, we may choose $1_{RQA} = 1_A$ to be $1'_A$ for any A, so $G(1_A) = 1_{F(A)}$ rather instantly. Hence, G is indeed a functor.

Consider a morphism $h : A \to B$ in $\mathscr{M}$ and some other $f : A \to B$ in $\mathbf{Ho}(\mathscr{M})$ such that $f = \gamma(h)$. We may choose a representative $f' : RQA \to RQB$ such that the diagram

$$\begin{CD} A @<{p_A}<< QA @>{i_{QA}}>> RQA \\ @V{h}VV @V{\tilde{h}}VV @VV{f'}V \\ B @<{p_B}<< QB @>{i_{QB}}>> RQB \end{CD}$$

commutes. If we apply F to this diagram, we see that $F(h) = F(p_B) \circ F(i_{QB})^{-1} \circ F(f') \circ F(i_{QA}) \circ F(p_A)^{-1} = G(f) = (G \circ \gamma)(h)$, so $F = G \circ \gamma$.

To see uniqueness, for any G' such that $G' \circ \gamma = F$, note that each morphism in $\mathbf{Ho}(\mathscr{M})$ of the form $\gamma(f)$ must be sent to $F(f)$ by G' and each $\gamma(f)^{-1}$ to $F(f)^{-1}$. Since all elements of $\mathbf{Ho}(\mathscr{M})$ are paths of these two kinds of morphism by Proposition 3.14, G' is uniquely determined so $G' = G$. □

We conclude this subsection with a satisfying observation, showing that the natural notion of homotopy between a cofibrant A and fibrant B is indeed exactly preserved in a homotopy category.

Proposition 3.16. [15, pg. 28] Suppose A is cofibrant and B is fibrant. Then $\gamma : \mathrm{Hom}(A, B) \to \mathrm{Hom}_{\mathbf{Ho}(\mathscr{M})}(A, B)$ is surjective and induces a bijection $\pi(A, B) \to \mathrm{Hom}_{\mathbf{Ho}(\mathscr{M})}(A, B)$.

Proof. We know that γ identifies homotopic maps, so the induced map $\pi(A, B) \to \mathrm{Hom}_{\mathbf{Ho}(\mathscr{M})}(A, B)$ is well-defined. To show surjectivity and bijectivity, consider the commutative diagram

$$\begin{CD} \pi(RA, QB) @>>> \pi(A, B) \\ @V{\gamma}VV @VV{\gamma}V \\ \mathrm{Hom}_{\mathbf{Ho}(\mathscr{M})}(RA, QB) @>>> \mathrm{Hom}_{\mathbf{Ho}(\mathscr{M})}(A, B) \end{CD}$$

where the horizontal arrows are induced by $[f] \mapsto [p_B \circ f \circ i_A]$ and commutativity is implied by γ being functorial. This means the lower map is pre/postcomposition with isomorphisms in $\mathbf{Ho}(\mathscr{M})$, so the bottom horizontal map is a bijection. RA and QB are both fibrant-cofibrant, so the leftmost vertical map is a bijection by construction. The uppermost horizontal map is also a bijection by Propositions 3.5 and 3.11. With this, the result is immediate. □

3.2.3 Homotopy Limits and Colimits

The reader may perhaps feel somewhat perplexed that we have seeminngly wound up at the same place that we started, having shown a model category to be a convoluted presentation of a localization. However, restricting our attention to $\mathbf{Ho}(\mathscr{M})$

for any model category $\mathscr{M}$ is a major disservice to the nuanced theory we have so far developed. Indeed, while it does serve to make entirely explicit what an 'inverse' to a weak equivalence $f \in W$ should be, namely a homotopy inverse after taking fibrant-cofibrant replacements, we are in fact able to extend limits and colimits to this more generalized homotopy-theoretic context.

Consider, for a model category $\mathscr{M}$ and small category $\mathscr{D}$, the familiar adjunction

$$\operatorname{colim} : \mathscr{M}^{\mathscr{D}} \Leftrightarrow \mathscr{M} : \Delta$$

formed by colimits and diagonals. We would like to consider a colimit *up to weak equivalence*, that is, the weak equivalence class of colimits we get from weak equivalence classes of elements in $\mathscr{M}^{\mathscr{D}}$.

Unfortunately, we will usually not get a weak equivalence class of colimits if we are too careless. Taking an example from [15, pg. 46], consider the model category **Top**, whose weak equivalences W are the homotopy equivalences. We may construct a diagram

$$\begin{array}{ccccc} D^n & \xleftarrow{j_n} & S^{n-1} & \xrightarrow{j_n} & D^n \\ \downarrow & & {\scriptstyle 1_{S^{n-1}}}\downarrow & & \downarrow \\ * & \longleftarrow & S^{n-1} & \longrightarrow & * \end{array}$$

where $j_n : S^{n-1} \to D^n$ is the inclusion of the boundary sphere and all vertical morphisms are homotopy equivalences. Observe that the pushout of the uppermost horizontal span is homeomorphic to S^n, while that of the bottom row is $*$, which are not weakly equivalent for all n. Hence, no well-defined colimit functor could exist between $\mathbf{Top}[W^{-1}]$ and $\mathbf{Fun}(\{a \leftarrow b \to c\}, \mathbf{Top})[W^{-1}_{\{a \leftarrow b \to c\}}]$.

Left and right derived functors will let us correct this issue. In essence, if we can show $\mathscr{M}^{\mathscr{D}}$ to have a model category structure, then we can build a universal approximation of the above adjunction

$$\mathbb{L}\operatorname{colim} : \mathbf{Ho}(\mathscr{M}^{\mathscr{D}}) \Leftrightarrow \mathbf{Ho}(\mathscr{M}) : \mathbb{R}\Delta$$

It is often the case that $\mathscr{M}^{\mathscr{D}}$ will have such a structure, such as in the cases of pushouts and pullbacks. This adjunction will of course merely be the idealized approximation of colimits or limits up to weak equivalence, since a direct attempt at this would fail as we have seen.

3.2.3.1 Derived Functors

We must first develop the theory of derived functors to a suitable degree. Some proofs will be omitted from this section.

Definition 3.11. [15, pg. 40-41] Suppose $\mathscr{M}$ is a model category and $F : \mathscr{M} \to \mathscr{C}$ is a functor. Then a *left derived functor* is a pair $(\mathbb{L}F, t)$ of a functor $\mathbb{L}F : \mathbf{Ho}(\mathscr{M}) \to \mathscr{C}$ and a natural transformation $t : \mathbb{L}F \circ \gamma \to F$ such that if (G, s) is any other such pair, then there exists a unique natural transformation $s' : G \to \mathbb{L}F$ such that the composition

$$G \circ \gamma \overset{s' \circ \gamma}{\to} \mathbb{L}F \circ \gamma \overset{t}{\to} F$$

is equal to s. A *right derived functor* is then another such pair $(\mathbb{R}F, t)$, this time with $t : F \to \mathbb{R}F \circ \gamma$ universal with regards to transformations $\mathbb{R}F \to G$ instead.

The pairs (G, s) above are intuitively approximations of F up to weak equivalences. The left and right derived functors $\mathbb{L}F$ and $\mathbb{R}F$ are then the approximations which all others must factor through, either via transformations *to* or *from* F respectively. A left or right derived functor is of course unique up to unique natural isomorphism by universality.

The reader familiar with homological algebra may be somewhat irritated at our reuse of the terminology 'derived', which otherwise means the extension of a functor to homology of injective or projective resolutions. However, this is no error. In fact, chain complexes admit a natural model structure with quasi-isomorphisms as their weak equivalences [15, ch. 7] and in this context, the derived functors are indeed well-named [15, pg. 43].

Proposition 3.17. [15, pg. 41] Let $F : \mathscr{M} \to \mathscr{C}$ be a functor sending weak equivalences between cofibrant objects to isomorphisms. Then the left derived functor $(\mathbb{L}F, t)$ exists and for each cofibrant object X of $\mathscr{M}$, the map $t_X : \mathbb{L}F(X) \to F(X)$ is an isomorphism.

We will not prove this result here. Regardless, we now have strong evidence for the existence of these derived functors in a variety of scenarios. This will become handy as we completely restrain ourselves to the context of homotopy categories:

Definition 3.12. [15, pg. 42] For a functor between model categories $F : \mathscr{M} \to \mathscr{N}$, a *total left derived functor* is a functor $\mathbb{L}F : \mathbf{Ho}(\mathscr{M}) \to \mathbf{Ho}(\mathscr{N})$ which is the left derived functor to $\gamma_{\mathscr{N}} \circ F : \mathscr{M} \to \mathbf{Ho}(\mathscr{N})$. A *total right derived functor* $\mathbb{R}F$ is similar.

What is noteworthy is that, should F preserve weak equivalences even just for cofibrant objects, then a left derived functor will always exist. This can be taken even further in a startling way:

Theorem 3.2. [15, pg. 43] Let $\mathscr{M}$ and $\mathscr{N}$ be model categories and

$$F : \mathscr{M} \Leftrightarrow \mathscr{N} : G$$

an adjunction. If F preserves cofibrations and G preserves fibrations, then the total derived functors

$$\mathbb{L}F : \mathbf{Ho}(\mathscr{M}) \Leftrightarrow \mathbf{Ho}(\mathscr{N}) : \mathbb{R}G$$

exist and form an adjunction. Furthermore, if for each cofibrant object A in $\mathscr{M}$ and fibrant object B in $\mathscr{N}$ we have that a map $f : A \to G(B)$ is a weak equivalence in $\mathscr{M}$ if and only if its adjoint $f' : F(A) \to B$ is a weak equivalence in $\mathscr{N}$, then $\mathbb{L}F$ and $\mathbb{R}G$ are mutually inverse equivalences of categories.

The proof is somewhat gargantuan, so we will avoid recalling it here; the reader is referred to [15, pg. 43-45]. An adjoint pair F, G such that F preserves fibrations and G cofibrations is called a *Quillen pair* [48, pg. 130]. We will refer to the left and right functors in such an adjunction as a *left Quillen functor* and *right Quillen functor*, respectively. Such a pair is further called a *Quillen equivalence* if the adjoint bijection of hom-sets is a bijection of weak equivalences, as in the above theorem.

To better cement that this universal approximation of a Quillen pair in homotopy categories is meaningful, we should take a step back and try to prove what the derived functors in such a scenario actually look like. In particular, if $\mathbb{L}F$ can be proven to satisfy the assumptions of Proposition 3.17, then it must be the same as F on cofibrant objects up to weak equivalence, which due to cofibrant replacements implies it must be the same on all objects up to weak equivalence. A formally dual result must then hold of $\mathbb{R}G$.

Lemma 3.7. [15, pg. 43-44] For any Quillen pair $F : \mathscr{M} \Leftrightarrow \mathscr{N} : G$, G preserves fibrations and acyclic fibrations. Dually, F preserves cofibrations and acyclic cofibrations.

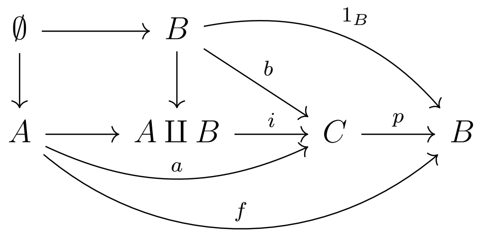
Proof. We prove here that G preserving fibrations implies that F preserves acyclic cofibrations. Suppose that $f : A \to B$ is an acyclic cofibration in $\mathscr{M}$ and $g : X \to Y$ a fibration in $\mathscr{N}$. Consider the left diagram and its adjoint diagram on the right:

$$\begin{array}{ccc} A & \xrightarrow{u} & G(X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle G(g)} \\ B & \xrightarrow{v} & G(Y) \end{array} \qquad \begin{array}{ccc} F(A) & \xrightarrow{u'} & X \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle g} \\ F(B) & \xrightarrow{v'} & Y \end{array}$$

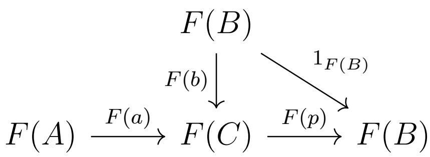
Since G preserves fibrations, there exists a lift $w : B \to G(X)$ in the left-hand diagram. This has an adjoint map $w' : F(B) \to X$, which must commute in the right-hand diagram by naturality and therefore be a lift. We now see that $F(f)$ has the LLP with regards to all fibrations, making it an acyclic cofibration. The dual proposition has a similarly dual proof. $\square$

Lemma 3.8. (Reedy lemma) [48, pg. 129] Let $F : \mathscr{M} \to \mathscr{N}$ be a functor between model categories. If F takes acyclic cofibrations to weak equivalences, then it takes weak equivalences between cofibrant objects to weak equivalences. Dually, if F takes acyclic fibrations to weak equivalences, then it takes weak equivalences between fibrant objects to weak equivalences.

Proof. We will prove the first statement, as the latter is dual. Let $f : A \to B$ be a weak equivalence between cofibrant objects in $\mathscr{M}$. This gives us a commutative diagram



where the square is a pushout and we've chosen a factorization of $(f, 1_B) : A \amalg B \to B$ as $p \circ i$, a cofibration followed by an acyclic fibration. Because A and B are cofibrant, the maps $A \to A \amalg B \leftarrow B$ are cofibrations by closure under cobase change. Moreover, this and the 2-out-of-3 property for weak equivalences imply that a and b are in fact acyclic cofibrations. Applying F now produces the diagram



where $F(b)$ and $F(a)$ are weak equivalences by hypothesis, $F(p)$ is a weak equivalence by 2-out-of-3 and $F(f) = F(p) \circ F(a)$ is then a weak equivalence for the same reason. $\square$

We now realize that for a Quillen pair $F : \mathscr{M} \Leftrightarrow \mathscr{N} : G$, F will preserve weak equivalences between cofibrant objects and G the same between fibrant ones. Importantly, $\gamma_{\mathscr{N}} \circ F$ satisfies the assumptions of Proposition 3.17, so $\mathbb{L}F$ will be identical to F on objects up to isomorphism in homotopy categories. Hence, colimit functors that form a Quillen pair will be invariant under weak equivalence on cofibrant replacements. Naturally, a dual result then holds for limits. It is therefore in looking for suitable such Quillen pairs that we will be able to develop homotopy limits and colimits.

3.2.3.2 Good Limits and Colimits

It is unfortunate that in general, for a small category $\mathscr{D}$ and model category $\mathscr{M}$, $\mathscr{M}^{\mathscr{D}}$ does not admit an obvious model structure [15, pg. 51]. It is our aim here to discover in what cases a suitable model structure exists, such that colimits and limits induce Quillen pairs. The cases where we succeed will be rather uninventively dubbed *good limits* and *good colimits* [48, pg. 130].

A straightforward example arises by setting $\mathscr{D}$ to be a small discrete category [48, pg. 130]. We may then say some morphism $\alpha : X \to X'$ in $\mathscr{M}^{\mathscr{D}}$ is a fibration, cofibration or weak equivalence when every $\alpha_i : X(i) \to X'(i)$ is as such [48, pg. 130]. This rather immediately forms a model structure, being nothing more than $|\mathscr{D}|$ parallel copies of $\mathscr{M}$. Again rather immediately, the colimit functor $\operatorname{colim} : \mathscr{M}^{\mathscr{D}} \to \mathscr{M}$ and the constant diagram functor $\Delta : \mathscr{M} \to \mathscr{M}^{\mathscr{D}}$ are adjoint *a priori* such that colim preserves cofibrations and Δ fibrations. The latter is by definition, while the former is due to cofibrations being closed under coproduct. This means small coproducts are good colimits.

Another example from [48, pg. 130] is that of the category ω, where $\mathrm{Ob}(\omega) = \mathbb{N}$ and $\mathrm{Hom}_\omega(m, n)$ contains one element if $m \leq n$ and is empty otherwise, ie. the ordered set $\mathbb{N}$:

$$1 \to 2 \to 3 \to \dots$$

For a model category $\mathscr{M}$, a morphism $\alpha : X \to X'$ in $\mathscr{M}^\omega$ may be called a weak equivalence or fibration if each α_i is as such, which uniquely defines the cofibrations. This does give a model category structure and induce a Quillen pair via the colimit and constant functors [48, pg. 131]. Hence, this constitutes another good colimit.

One final form of colimit we consider here is a pushout. For the diagram $\mathscr{D} = \{1 \leftarrow 0 \to 2\}$, we say an morphism $\alpha : X \to X'$ of $\mathscr{M}^{\mathscr{D}}$ is a weak equivalence if it is objectwise, a cofibration if α_0, α_1 and $X(2) \amalg_{X(0)} X'(0) \to X'(2)$ are all cofibrations, and a fibration if α_1, α_2 and the map $X(1) \to X'(1) \times_{X'(0)} X(0)$ are fibrations. This again forms a suitable model structure and a Quillen pair via the colimit functor [48, pg. 131], forming a final good colimit.

One should note that this last good colimit directly resolves the pathological example of pushouts we considered. Indeed, we now see that the pushout will be invariant up to weak equivalence on cofibrant objects in $\mathscr{M}^{\mathscr{D}}$, rather than on all possible diagrams. Again, fibrant and cofibrant replacements rectify the pathological behaviors of general weak equivalences.

All these good colimits allow for a derived functor $\mathbb{L}$colim, which by our earlier observations is identical to colim up to isomorphism in $\mathbf{Ho}(\mathscr{M})$. Dual arguments give us identical results for the limits of opposite diagrams to above, including products and pullbacks [48, pg. 131].

It is in fact possible to go a bit further than the good limits and colimits we have considered; [15, pg. 50] discusses the so called *very small* categories $\mathscr{D}$, which have a finite number of objects and morphisms and some $n \in \mathbb{N}$ such that chains of morphisms longer than n compose to give an identity. It turns out that these all admit two model structures adapted to constructing $\mathbb{L}$colim and $\mathbb{R}$lim respectively, both with the same weak equivalences [15, pg. 50] and therefore essentially the same.

One warning we recall is that, as noted in our discussions surrounding localization, $\mathbf{Ho}(\mathscr{M}^{\mathscr{D}})$ and $\mathbf{Ho}(\mathscr{M})^{\mathscr{D}}$ are in general not equivalent [15, pg. 47]. This is to say that homotopy limits are not limits in the homotopy category - rather, they are limits in the model category up to homotopy. As noted by Dwyer and Spalinski, "... much of the subtlety of homotopy theory lies in this fact." [15, pg. 47]

3.2.4 Left Bousfield Localizations

An important consideration in model category theory is how one may add some class H of morphisms to the weak equivalences a given model category has. Clearly this is not always possible due to the potential collapse of the 2-out-of-3 property or the weak factorization systems involved, though a universal approximation may be feasible:

Definition 3.13. [4, pg. 42] Suppose $\mathscr{M}$ is a model category[1] and H is a collection of homotopy classes of morphisms in $\mathscr{M}$. A *left Bousfield localization* of $\mathscr{M}$ with respect to H is a model category $L_H\mathscr{M}$, equipped with a left Quillen functor $\mathscr{M} \to L_H\mathscr{M}$ that is initial amongst all other left Quillen functors $F : \mathscr{M} \to \mathscr{N}$ to model categories $\mathscr{N}$ where for any f representing a class in H, $\mathbb{L}F(f)$ is an isomorphism in $\mathbf{Ho}(\mathscr{N})$.

Hence, we should think of $L_H\mathscr{M}$ as the universal model category where all the morphisms representing homotopy classes in H became new weak equivalences. If we assume some technical properties of our model category and set of homotopy classes,

[1]In the actual definition, as seems to be the case with many others in this dissertation regarding model categories, a Grothendieck universe is explicitly assumed. I abstain from such explicit decisions on foundations here, in the name of abstraction.

it turns out this category has some reasonable structure whose proof goes beyond the scope of this dissertation.

We will call a model category $\mathscr{M}$ *left proper* if its weak equivalences are preserved by pushouts along cofibrations [47, pg. 3] and *combinatorial* if it is locally presentable and *cofibrantly generated*, meaning some small set of cofibrations generates all the cofibrations in $\mathscr{M}$, which we also demand for acyclic cofibrations [33, pg. 210] [4, pg. 11]. We will not delve into the substance of these properties here, though may implicitly assume them to be the case when we perform left Bousfield localizations.

Proposition 3.18. [4, pg. 43] Suppose $\mathscr{M}$ is a left proper combinatorial model category and H is a small set of homotopy classes of morphisms in $\mathscr{M}$. Then $L_H\mathscr{M}$ exists. Moreover, $L_H\mathscr{M}$ and $\mathscr{M}$ have the same underlying category and share the same cofibrations.

We lack time to discuss exactly what the resulting weak equivalences are in $L_H\mathscr{M}$, though it contains H. We thus see that in most cases, this construction simply modifies the weak equivalences and fibrations of the same category. The interested reader is directed to [4] for details.

3.2.5 Why Model Categories Fall Short

It seems model category theory solves a great deal of problems with just localization, containing enough information to consider limits and colimits up to the chosen notion of equivalence. The reader may at this point be inclined to convert our notion of a stack in groupoids to one in model categories, demanding our pullback functors be left Quillen or some similar condition. We may then try to design a model category of stacks over some site, drawing all these weak equivalence structures out of any one stack and into the relationships between stacks, as category theory urges us to do.

While we will indeed use some form of this construction later, it turns out model categories are not yet sufficient to satisfy all our needs. For starters, given two model categories $\mathscr{M}$ and $\mathscr{N}$, there is no completely general model structure on the functor category $\mathbf{Fun}(\mathscr{M}, \mathscr{N})$ [61, pg. 12]. This will make our construction of a category of stacks somewhat non-functorial, as well as potentially adding difficulty when designing a stack of morphisms between these hypothetical stacks.

Another issue strikes us as we consider the difficulties in, given some class of weak equivalences, choosing suitable fibrations and cofibrations. These classes of morphism are purely a technicality introduced to better understand and manipulate the weak equivalences, so they should not get in our way. Unfortunately, there is no generalized procedure by which one may produce a model structure given some choice of weak equivalences - if there were, the functor category would not be so difficult to define for instance, as the weak equivalences should clearly be those natural transformations that are as such object-wise. Truly, we find ourselves ensnared by technicalities to an unnecessary degree.

Furthermore, while we can understand when two objects are weakly equivalent, we are currently somewhat incapable of understanding how exactly two morphisms are made the same. For instance, consider the classical case of the model category representing homotopies in **Top**. What if we are interested in studying the actual homotopies themselves? We would have to step outside our current model category theory. As a rather different example, consider a groupoid generated by an algebraic group action X/G. If $N \lhd G$ is a normal subgroup, we may be interested in understanding the induced action $X/(G/N)$. To do this, we could add 'higher morphisms' between any two morphisms in X/G, representing right multiplication by elements in N. Such a construction would not only produce the quotient group action by taking 'higher isomorphism classes' of morphisms in X/G, but would let us study exactly how the quotient modifies the action explicitly. Unfortunately, model category theory doesn't suit this kind of construction. Even worse, every morphism in X/G is already an isomorphism, so any class of weak equivalences would include everything!

Was all our work up to this point a waste? We urge the reader not to lose hope, as we reveal our actual motivation all along: model categories merely *present* the true, intrinsic object we are searching for. Indeed, the homotopy categories we have considered thus far may represent homotopy classes of morphisms, but lose all the information about the homotopies that identify two morphisms, the homotopies that identify two different homotopies, *ad infinitum*. If an object existed that contained all these higher homotopies for some model category $\mathscr{M}$ and weak equivalences W, say $L(\mathscr{M}, W)$, it could be said, informally at this stage, that

$$\mathbf{Ho}(\mathscr{M}) = \pi_0 L(\mathscr{M}, W)$$

ie. that the homotopy category is just the path components of morphisms in this more nuanced construction.

Indeed, it will turn out that a construction of this form is the one we seek, eventually comprising an internal theory should we define it correctly. Cofibrations and fibrations will fall to the wayside and almost any homotopical invariant in a model category may be reconstructed from this enigmatic $L(\mathscr{M}, W)$ [61, pg. 14].

3.3 Dérivateurs

Before we dive into developing the supposed magic bullet we hinted at in the previous section, we pay lip service to another approach to fixing the issues with localization, suggested separately by Heller and Grothendieck [61, pg. 13]. Taking a different approach to model categories are the so-called *dérivateurs*, namely the categories of all functor categories $\mathbf{Fun}(I, \mathscr{C})[W_I^{-1}]$ for all I. Consider the functor $\mathbb{D}_{(\mathscr{C},W)}$: $\mathbf{Cat}^{op} \to \mathbf{Cat}$ defined as[2]

$$
\begin{aligned}
I &\mapsto \mathbf{Fun}(I, \mathscr{C})[W_I^{-1}] \\
f &\mapsto f^*
\end{aligned}
$$

where f^* is precomposition. This is in fact a so-called *2-functor*, one which also preserves the natural transformations in $\mathbf{Cat}^{op}$ in a natural way [61, pg. 13]. We then get a category of 2-functors $\mathbf{Cat}^{op} \to \mathbf{Cat}$ of this form with natural transformations, called **PDer**, the category of *pré-dérivateurs* [22, pg. 5]. In fact, given two morphisms in **PDer**, we may have a higher transformation between them along the natural transformations in **Cat**, making **PDer** a *2-category.* The technicalities of this definition are not so important here; for now, the reader may consider how natural transformations act as 'higher morphisms' between morphisms in **Cat**.

Some extra axioms detailed by Groth in [22, pg. 10] restrict us to the *dérivateurs*, which we will not explore in depth. The hope is that these constructions, tracking all the information explicitly that localizations forget, will resolve the problems with localization on its own, letting us form limits, colimits and mapping spaces up to the weak equivalences as we please. Indeed, this theory is rather clearly internal and resolves many issues in far-reaching subjects like derived algebraic geometry [61, pg. 13]. Unfortunately, it turns out that dérivateurs are merely a coarse approximation up to 2-homotopies of the more complex objects we will soon turn our attention to [61, pg. 14, 28], forgetting what are in essence the 'higher automorphisms' and artificially barring us from their study. Moreover, the various homotopy invariants that may crop up in a given model category may require the investigation of higher homotopies [61, pg. 14], meaning we must look deeper for a more complete construction.

[2] There is an implicit assumption in this definition that $\mathbf{Fun}(I, \mathscr{C})[W_I^{-1}]$ is small so it may truly reside in **Cat**, even though $\mathscr{C}$ is somewhat unrestricted. We ignore this technicality here.

3.4 Simplicial Structures

As emphasized in our closing discussions of model categories, we should not see a model category as our intrinsic object of study owing to its lack of internal structure and obfuscatory technical extremities, but rather as a presentation of something more fundamental. This fundamental structure should somehow augment our category $\mathscr{C}$ and class of weak equivalences W with *higher morphisms*, dictating the homotopies that identify weak equivalences, the homotopies identifying these and so on. Taking path components of such a structure would give us back our localizations, or homotopy categories, while taking path components at higher levels would give us information about the homotopies at each level. Furthermore, this structure would not necessarily need the notion of a fibration or cofibration, hopefully freeing us of this technical burden.

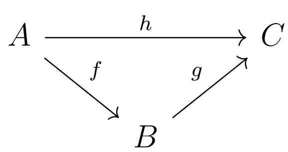
Note that if we were ever to stop taking higher homotopies, say up to some level $n \in \mathbb{N}$, the category of 'functors' between two such structures would have n levels of higher homotopies itself, ie. natural transformations at each level of homotopy. This is clear in the case where $n = 1$, where there are merely objects and morphisms as normal; the 1-morphisms in $\mathbf{Fun}(\mathscr{C}, \mathscr{D})$ are natural transformations along morphisms. Hence, the category of all such structures would require homotopies up to at least level $n + 1$. This means our theory would not be self-contained, obliging us to set $n = \infty$ at minimum.

Such a goal will lead us to define the *simplicial localization*, a structure developed by Dwyer and Kan in 1979 [13]. Given a category $\mathscr{C}$ and class of morphisms W, this structure, written as $L(\mathscr{C}, W)$, adds higher homotopies to $\mathscr{C}$ such that taking homotopy path components exactly produces $\mathscr{C}[W^{-1}]$. This structure, with some extra fine-tuning, will yield all the sensible properties we seek.

3.4.1 Simplicial Sets

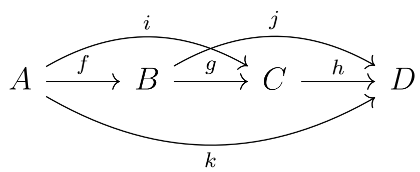
Before we can begin defining such a structure, we need to formalize what we mean by a 'higher morphism'. Consider a small category $\mathscr{C}$. As a simple exercise, we will try to generate higher morphisms that represent the compositions in $\mathscr{C}$.

Given a commutative diagram of the form

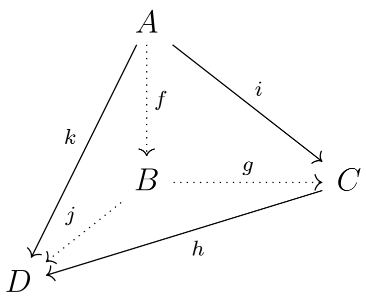


we would like to introduce a 'higher morphism' from f and g to h. Geometrically, if we chose to see $\mathscr{C}$ as the topological realization of its graph, ie. a collection of path intervals $\mathrm{Mor}(\mathscr{C})$ joined at points $\mathrm{Ob}(\mathscr{C})$, we could fill the empty space between the paths f, g and h with a filled 2-dimensional triangle, or the *2-simplex* Δ^2. Once this is done, we could say that 'f and g compose to give h' precisely when there is a deformation retract of the above diagram to just the path h.

Defining composition this way leads us to fill a number of diagrams of the above form with 2-simplices, letting us observe via these higher morphisms when two morphisms compose to give another. Unfortunately, we arrive at an impasse when considering larger compositions:



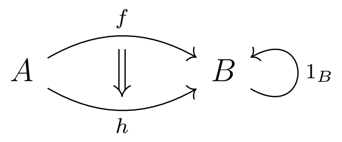
The picture may become a bit clearer when we rearrange it into a tetrahedron:



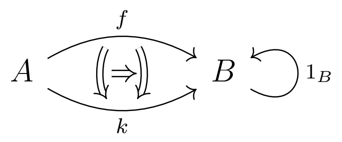
The morphism k, identified on the upper left, must be the unique end result of all paths $A \to D$. The 2-simplices we glue to show compositions of pairs of morphisms will form the faces of this tetrahedron. However, there is no deformation retract of the above diagram to the path k, owing to the empty space inside the tetrahedron. How are we to deal with this discrepancy?

One solution is to fill the space inside the diagram with a filled tetrahedron, or *3-simplex* Δ^3. With this, demanding associativity for sequences of n morphisms simply requires filling an n-dimensional tetrahedron with an n-simplex Δ^n.

Why would this generalize to a good notion of higher homotopy? At a glance, we are only playing with the so-called 1-morphisms and their compositions. Furthermore, the 2-simplices for instance don't quite seem to be representing the 2-morphisms between two arrows f and h, but rather include a third one g, which would be a spurious addition in other contexts. However, reconsider the 2-dimensional case where we set $B = C$ and $g = 1_B$. Now, the 2-simplex filling the diagram is a higher morphism from f to h, identifying them via composition with 1_B:



Similarly, we may reconsider the 3-dimensional case where $B = C = D$ and $g = h = 1_B$. We now see that the 3-simplex filling the tetrahedron constitutes a 3-morphism between 2-morphisms from f to k:



Generalizing to higher dimensions shows how an n-simplex may be seen inductively as an n-morphism between $(n-1)$-morphisms. Hence, filling in triangles of morphisms rather than pairs sharing source and target is in fact a more general version of what we desired, letting us not only consider the higher morphisms we expected but also higher morphisms between entire paths of morphisms directly. Such flexibility grants us access to, for example, directly defining a notion of homotopy inverse, by changing which morphisms above are identities.

This seems to suggest a natural notion of higher category, namely a collection of objects and morphisms between them, together with some n-simplices for all n, whose vertices are the objects and edges are the morphisms. Note however that a vertex is just a 0-simplex and a morphism a 1-simplex, so our new structure is merely 'a collection of simplices that may share faces'.

Perhaps the ideal way to gather all this information is not through a literal topological space, but by simply tracking all the simplices and how they are interconnected. This will let us consider the structure present in a more measured manner, maintaining an explicit awareness of which simplices are attached where, along with the all-too-important orientation of the paths. If we rephrase 'simplex' as 'generalized Δ^n-point', our intuitions about sheaves lead us to realize that this challenge is exactly what presheaves are for. A presheaf of this form will be called a *simplicial set*, taking inspiration from simplicial complexes, and will form the foundation for our concept of higher morphisms going forward.

We should declare a category containing the types of generalized points we want, namely the n-simplices, along with morphisms $\Delta^n \to \Delta^m$ indicating when one resides within another. These morphisms should send vertices to vertices, edges to edges and so on. Furthermore, they should preserve *orientation*, as our morphisms are indeed oriented, so the edges of these simplices will be as well. With all this in mind, we may dramatically simplify things by replacing Δ^n with the ordered set $\overline{n} = \{0, 1, \ldots, n\}$, ie. its ordered set of vertices, and all maps $\Delta^n \to \Delta^m$ with monotonic maps $\overline{n} \to \overline{m}$, ie. maps between vertex sets respecting the orientation of morphisms.

Definition 3.14. [48, pg. 7] The *simplicial indexing category*, written as Δ, is the category whose objects are totally ordered sets $\overline{n}$ for all $n \in \mathbb{N} \cup \{0\}$ and whose morphisms are order-preserving maps.

We will often write a morphism f in Δ as

$$f = \langle f_0, f_1, \ldots, f_n \rangle : \overline{n} \to \overline{m},\ f_0 \leq \cdots \leq f_n$$

for the morphism mapping $k \mapsto f_k$ for $0 \leq k \leq n$ [48, pg. 8].

It is noted in [48, pg. 8] that every morphism in Δ can be replaced with a composition of some *face* and *degeneracy* operators d^i and s^i respectively, ie. maps of the form

$$d^i = \langle 0, \ldots, i-1, i+1, \ldots, n \rangle : \overline{n-1} \to \overline{n}$$
$$s^i = \langle 0, \ldots, i, i, \ldots, n \rangle : \overline{n+1} \to \overline{n}$$

The map d^i maps an $(n-1)$-simplex onto the face of an n-dimensional simplex not containing the vertex i, while s^i degenerately compresses an $(n+1)$-simplex to one dimension lower by identifying the vertices i and $i+1$.

We now come to the purpose of this category, namely to use it for the presheaves we seek:

Definition 3.15. [48, pg. 8] A *simplicial set* is a presheaf on Δ.

For a simplicial set X, we will often write X_n in favor of $X(\overline{n})$, which will be called the set of n-simplices of X. Clearly, the natural transformations of simplicial sets $X \to Y$ will simply be mappings of n-simplices to n-simplices $X_n \to Y_n$ respecting the faces $(d^i)^*(x)$ of each n-simplex x.

Definition 3.16. [48, pg. 8] The *category of simplicial sets*, denoted **sSet**, is the category $\mathbf{Fun}(\Delta^{op}, \mathbf{Set})$.

It is not hard to define some simple simplicial sets, the first of which will realize our current informal notion of an n-simplex in this context:

Definition 3.17. [48, pg. 9] The *standard n-simplex*, written Δ^n, is the presheaf represented by $\overline{n}$. More explicitly, it is the presheaf whose sets of simplices are

$$\Delta^n_i = \mathrm{Hom}(\overline{i}, \overline{n})$$

and whose pullback morphisms are defined by precomposition.

It is not hard to see that this is indeed what we expected a standard n-simplex to be, owing to our intuitions about presheaves as collections of generalized points. As always, the Yoneda lemma lets us identify the n-simplices of a simplicial set X, namely the elements of X_n, with the natural transformations $\Delta^n \to X$.

Another somewhat trivial example is that of a *discrete simplicial set*. Imagine if every $f : \overline{m} \to \overline{n}$ induced a bijection $f^* : X_n \to X_m$ in X. In such a case, the maps $\langle 0 \rangle : \overline{0} \to \overline{n}$ for all n would imply that X is uniquely identified by X_0. Conversely, any set S corresponds to a simplicial set X where $X_n = S$ and all pullback morphisms are identities. Any map between such simplicial sets $X \to Y$ could then be reduced to the function $X_0 \to Y_0$ and vice versa. Hence, we have the following:

Definition 3.18. [48, pg. 9] A *discrete simplicial set X* is one where every pullback morphism is a bijection.

Note that every n-simplex in a discrete simplicial set for $n > 0$ is in the image of the degeneracy operators $(d^i)^*$ for each i, meaning we should see them as the degenerate interpretation of a lower-dimensional simplex as a higher-dimensional one. This suggests the visual of a set of disconnected points, any one of which could be interpreted as a trivial line or tetrahedron where everything is at the same point. Indeed, the fact that a map between discrete simplicial sets is determined entirely by its behavior on the 0-simplices suggests that no such higher simplex can be truly distinguished from its only vertex.

Such insight implies a definition we will find useful later:

Definition 3.19. [48, pg. 38] Given a simplicial set X, an n-simplex $a \in X_n$ is called *degenerate* if there exists a non-identity surjection $f \in \mathrm{Mor}(\Delta)$ and m-simplex $b \in X_m$ such that $a = f^*(b)$.

We may of course simply demand $f : \overline{n} \to \overline{m}$ to be non-injective, as any morphism in Δ can be factored through a surjection $f_{surj} : \overline{n} \to \overline{p}$ and then an injection $f_{inj} : \overline{p} \to \overline{m}$ by letting $p = |f(\overline{n})|$ [48, pg. 9].

A more involved example of a simplicial set will be our case study of higher morphisms representing composition in a category. To do this, we consider the category $[\overline{n}]$ for any $n \in \mathbb{N}$, whose objects are $\overline{n}$ and morphisms are the total ordering.

Definition 3.20. [48, pg. 11] Given a small category $\mathscr{C}$, the *nerve* $N\mathscr{C}$ is the simplicial set whose n-simplices are of the form

$$N\mathscr{C}_n = \mathrm{Hom}_{\mathrm{Cat}}([\overline{n}], C)$$

and whose pullback morphisms are defined by precomposition.

If we think about this definition long enough, we realize that $N\mathscr{C}_n$ may be identified with the chains of n arrows in $\mathscr{C}$, with $N\mathscr{C}_0 = \mathrm{Ob}(\mathscr{C})$ and $N\mathscr{C}_1 = \mathrm{Mor}(\mathscr{C})$. Furthermore, for some $x \in N\mathscr{C}_n$ with $n > 1$, the morphisms $\langle i, i+1\rangle^*(x)$ all compose into $\langle 0, n\rangle^*(x)$, from the source $\langle 0\rangle^*(x)$ to the target $\langle n\rangle^*(x)$. Note that for any $y \in N\mathscr{C}_n$, if we have $\langle i, i+1\rangle^*(x) = \langle i, i+1\rangle^*(y)$ for all i then $x = y$, representing uniqueness of composition. Associativity is then implicit as we realize that x is also unique with regards to all the sub-compositions $\langle i, j\rangle^*(x)$.

There are a few constructions we would like to consider for simplicial sets that we need to formalize. The first we will develop is the path components of a simplicial set X.

Definition 3.21. [48, pg. 21] Given a simplicial set X, its *set of path components* $\pi_0 X$ is the quotient of the set $\amalg_{n\geq 0} X_n$ by the smallest equivalence relation $\cong$ identifying every $x \in X_n$ with all its images under pullback maps.

Intuitively, this construction identifies any simplex with its faces and vice versa. Another way to construct this is to consider the quotient of X_0 by the smallest relation $\sim$ identifying 0-simplices if they are both faces of the same 1-simplex, ie. $a \sim b$ if there is some $x \in X_1$ such that $(d^1)^*(x) = a$ and $(d^0)^*(x) = b$ [48, pg. 21]. This latter notion identifies vertices 'if there is a path between them'.

Proposition 3.19. [48, pg. 21] There is a canonical bijection from $X_0/\sim$ to $\pi_0 X$.

Proof. The canonical map $X_0/\sim \to \pi_0 X$ sending $[x]_\sim \mapsto [x]_\cong$ for $x \in X_0$ is clearly well-defined, as if $x \sim y$ via some $z \in X_1$, then $x \cong z \cong y$ by face operators. Surjectivity is rather straightforward; for any $x \in X_n$, we have $[x]_\cong = [\langle 0\rangle^*(x)]_\cong$, whose preimage is of course $[\langle 0\rangle^*(x)]_\sim$.

Injectivity is harder. If $x \cong y$ for $x, y \in X_0$ then either $x = y$ or there are some simplices z_i such that $x \cong z_0 \cong \ldots \cong z_r \cong y$, where each equivalence is directly induced by a pullback map. We can compose these aggressively to end up either in the situation $x \overset{f_0^*}{\to} z_0 \overset{f_1^*}{\leftarrow} z_1 \ldots y$ or $x \overset{f_0^*}{\leftarrow} z_0 \overset{f_1^*}{\to} z_1 \ldots y$ for $f_i \in \mathrm{Mor}(\Delta)$.

In the former case, we note that necessarily $f_0 = \langle 0, 0, \ldots, 0\rangle : \overline{n} \to \overline{0}$, so any $f : \overline{0} \to \overline{n}$ is such that $f_0 \circ f = 1_{\overline{0}}$. This means $(f_0 \circ f)^*(x) = x$, which of course implies $f^*(z_0) = x$. This lets us convert the former case into the latter by removing z_0 and replacing f_1 with $f_1 \circ f$.

We thus restrict ourselves only to the latter case. Consider f_0, f_1 such that $f_0^*(z_0) = x$ and $f_1^*(z_0) = z_1$. Moreover, let $f_0 = \langle j \rangle$ and $f_1 = \langle k_0, \dots \rangle$. Define the 0-simplex $p = \langle 0 \rangle^*(z_1)$ and the 1-simplex $t = \langle j, k_0 \rangle^*(z_0)$. Clearly $\langle 0 \rangle^*(t) = x$ and $\langle 1 \rangle^*(t) = p$, so $x \sim p$. We now can define a new chain $p \overset{\langle 0 \rangle^*}{\leftarrow} z_1 \dots y$ which is one element shorter. By induction, we get a chain $x \sim p \sim \cdots \sim y$ and so our map is injective, completing the proof. □

Another standard construction is that of a product of two simplicial sets, analogous to the geometric or category theoretic cases we have drawn parallels to. Of course, simplicial sets are just presheaves, so the product of two simplicial sets $X \times Y$ is just the simplicial set whose simplices are $(X \times Y)_n = X_n \times Y_n$ and whose pullback morphisms are the product of the respective maps in X and Y [48, pg. 20].

One interesting operation we will be able to perform is that of removing all non-degenerate m-simplices for all $m > n$. Such an operation amounts to 'truncating' a simplicial set, removing all higher-dimensional data. If we did this where $n = 1$, we would have nothing but points and lines, leaving behind a skeleton of sorts. This will motivate our terminology of a *skeleton* $\mathbf{sk}_n X$.

Definition 3.22. [48, pg. 40] The *k-skeleton* of a simplicial set X, denoted $\mathbf{sk}_k X$, is the minimal simplicial subset (levelwise) containing all non-degenerate j-simplices for $0 \leq j \leq k$. Explicitly, the skeleton takes the form

$$(\mathbf{sk}_k X)_n = \bigcup_{0 \leq j \leq n} \{f^*(y) \mid y \in X_j, f : \overline{n} \to \overline{j} \in \mathrm{Mor}(\Delta)\}$$

As an interesting aside, consider the full subcategory $\Delta^{\leq n}$ of Δ whose objects are those $\overline{k}$ such that $k \leq n$. The inclusion $\Delta^{\leq n} \to \Delta$ induces a forgetful functor $\mathrm{tr}_n : \mathbf{sSet} \to \mathbf{sSet}^{\leq n}$, where the latter category is just $\mathbf{Fun}((\Delta^{\leq n})^{op}, \mathbf{Set})$. This functor has a left adjoint $\mathrm{sk}_n : \mathbf{sSet}^{\leq n} \to \mathbf{sSet}$, artificially adding a unique element $f^*(x)$ for every $x \in X_m$, $f : \overline{k} \to \overline{m}$, $k > n \geq m$ such that compositions of operators are respected. Checking the adjoint condition

$$\mathrm{Hom}_{\mathbf{sSet}}(\mathrm{sk}_n(X), Y) \cong \mathrm{Hom}_{\mathbf{sSet}^{\leq n}}(X, \mathrm{tr}_n(Y))$$

is rather trivial, as a map between simplicial sets maps degenerate simplices somewhere uniquely determined by their source simplices. We may now see that $\mathbf{sk}_n X \cong (\mathrm{sk}_n \circ \mathrm{tr}_n)(X)$, so the skeleton operation is functorial.

Intriguingly, the forgetful functor tr_n has a *right adjoint* as well, of the form $\text{cosk}_n : \mathbf{sSet}^{\leq n} \to \mathbf{sSet}$. The condition

$$\text{Hom}_{\mathbf{sSet}}(X, \text{cosk}_n(Y)) \cong \text{Hom}_{\mathbf{sSet}^{\leq n}}(\text{tr}_n(X), Y)$$

implies that every non-degenerate m-simplex in X_m, for $m > n$, must uniquely map somewhere in $\text{cosk}_n(Y)_m$. Hence, this functor adds, for every ordered list of $n+1$ n-simplices, a unique $(n+1)$-simplex with those simplices as its faces in order. This proceeds inductively upwards for higher n. We write $\mathbf{cosk}_n = \text{cosk}_n \circ \text{tr}_n$ for the *coskeleton functor* [20, pg. 224]. It is indeed the case that $\mathbf{sk}_n$ is left adjoint to $\mathbf{cosk}_n$ for all n:

$$\text{Hom}_{\mathbf{sSet}}(\text{sk}_n(\text{tr}_n(X)), Y) \cong \text{Hom}_{\mathbf{sSet}^{\leq n}}(\text{tr}_n(X), \text{tr}_n(Y)) \cong \text{Hom}_{\mathbf{sSet}}(X, \text{cosk}_n(\text{tr}_n(Y)))$$

We will often write $\partial\Delta^n := \mathbf{sk}_{n-1}\Delta^n$ for convenience.

The final construction we will consider is more nuanced. We would like to consider the space of maps between two simplicial sets. It is clear that two maps $X \to Y$ could be 'homotopic', deforming along the simplicial structure. Such a homotopy is just a map $X \times \Delta^1 \to Y$. Indeed, we could consider higher dimensional homotopies $X \times \Delta^n \to Y$ for all n, demonstrating homotopies between homotopies and so on. If we do this, the space of maps $\Delta^0 \to X$ would be identical to X, a satisfyingly natural behavior that leads us to the following definition:

Definition 3.23. [48, pg. 32] Given two simplicial sets X and Y, we may form a *function complex* $\mathbf{Map}(X, Y)$, the simplicial set whose simplices are

$$\mathbf{Map}(X, Y)_n = \text{Hom}_{\mathbf{sSet}}(X \times \Delta^n, Y)$$

and whose pullback morphisms f^* for any $f : \overline{m} \to \overline{n}$ are of the form

$$\text{Hom}_{\mathbf{sSet}}(1_X \times f^*, Y) : \text{Hom}_{\mathbf{sSet}}(X \times \Delta^n, Y) \to \text{Hom}_{\mathbf{sSet}}(X \times \Delta^m, Y)$$

What motivates this form of pullback morphism? We may think of it as manipulating the structure of the homotopy itself; each Δ^n above has one vertex for each morphism $X \to Y$ and edges for homotopies, so something like $\langle 0 \rangle$ would extract the 0^{th} morphism, $\langle 0, 2 \rangle$ would extract the homotopy from the 0^{th} to the 2^{nd}, and so forth.

3.4.2 Model Structure on sSet

Before we finish with simplicial sets, we must observe a fundamental fact about them that we will unfortunately have no time to prove directly: there is a natural model structure on **sSet**.

In order to define this structure, we should begin with the weak equivalences. These should really be analogous to homotopy equivalences, so that we wind up with a homotopy theory of simplicial sets in a similar manner to topological spaces. Such a line of inquiry should be prefaced with a more concrete understanding of the relationship between **sSet** and **Top**.

Definition 3.24. [48, pg. 25] The *topological n-simplex* $\Delta^n_{top} \in$ **Top** is the space defined as

$$\Delta^n_{top} = \left\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \,\middle|\, \sum_{i=0}^{n} x_i = 1 \right\}$$

This is of course just the standard topological version of the n-simplices we have considered so far, in some sense a linear interpolation of Δ^n.

Proposition 3.20. [48, pg. 25] There is a functor Sing : **Top** → **sSet** such that

$$\mathrm{Sing}(X)_n = \mathrm{Hom}_{\mathbf{Top}}(\Delta^n_{top}, X)$$

with pullback functions defined by precomposition. Morphisms in **Top** are sent to the natural transformations defined by postcomposition.

The name here is short for 'singular', stemming from singular complexes in algebraic topology.

Proposition 3.21. [48, pg. 47] There is a functor $|\cdot|$: **sSet** → **Top** defined on objects as

$$|X| = \mathrm{Coeq}\left[\coprod_{f:\overline{m}\to\overline{n}} X_n \times \Delta^m_{top} \rightrightarrows \coprod_{p\in\mathbb{N}\cup\{0\}} X_p \times \Delta^p_{top} \right]$$

where Coeq means coequalizer and the two maps send $(x, y) \in X_n \times \Delta^m_{top}$ to $(f^*(x), y)$ and $(x, f'(y))$, where f is the map this point is indexed over and f' is the linear interpolation of f. The functor sends morphisms between simplicial sets $F : X \to Y$ to the induced maps on coequalizers by $\amalg_{f:\overline{m}\to\overline{n}}(F_n, 1_{\Delta^m_{top}})$ and $\amalg_{p\in\mathbb{N}\cup\{0\}}(F_p, 1_{\Delta^p_{top}})$.

This is a rather complex looking functor, but in reality all it is doing is producing a topological simplex Δ^p_{top} for each abstract p-simplex in X and gluing them together along the pullback morphisms f^*. The importance of this functor is emphasized by it being *left adjoint* to Sing [48, pg. 25], a somewhat laborious though straightforward fact to prove directly.

Furthermore, we have a guarantee that $|X|$ will always be a CW complex [48, pg. 25]. Though we haven't shown this formally here, it is somewhat clear that this will be the case. What matters is that, if we were to try to produce a homotopy theory of simplicial sets, we should now expect it to arise from the weak homotopy equivalence model structure on **Top**:

Theorem 3.3. (Kan, Quillen) [20, pg. 67] **sSet** admits a model structure whose weak equivalences are those morphisms sent to weak homotopy equivalences by $|\cdot|$ and whose cofibrations are the monomorphisms $X \to Y$, ie. level-wise injections $X_n \to Y_n$ for all $n \in \mathbb{N}$.

We will often return to this model structure or assume its presence. As an aside, the adjunction we have seen in fact induces an equivalence of categories between the homotopy categories **Ho**(**sSet**) and **Ho**(**Top**) [20, pg. 69].

3.4.3 Simplicially Enriched Categories

Somehow, we would like to produce some higher notion of a category from simplicial sets. The most obvious approach here would be to add constraints to a simplicial set that makes it suitably similar to a category, which is the approach of quasicategories [48]. There is, of course, a fundamental problem with this: simplicial sets are *small*. We could try to dance around this issue by using Grothendieck universes [48, pg. 23] but forcing ourselves to adopt exactly one formalism of category theory is poor form. Furthermore, the concept of composition is somewhat hard to define in this context directly.

To circumvent these issues, we note that while almost all categories we care for are not small, they are in fact locally small. Recall how abelian categories are *enriched* over abelian groups, meaning their hom-sets are abelian groups respected by composition. Perhaps enriching over simplicial sets will provide the higher categorical structure we seek:

Definition 3.25. [52, pg. 69] A *simplicial category* is a category $\mathscr{C}$ enriched in simplicial sets, meaning $\mathrm{Hom}_{\mathscr{C}}(A, B)$ is a simplicial set for every $A, B \in \mathrm{Ob}(\mathscr{C})$ and composition is a map of simplicial sets $\circ : \mathrm{Hom}_{\mathscr{C}}(A,B)\times\mathrm{Hom}_{\mathscr{C}}(B,C) \to \mathrm{Hom}_{\mathscr{C}}(A,C)$ such that the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathscr{C}}(A,B) \times \mathrm{Hom}_{\mathscr{C}}(B,C) \times \mathrm{Hom}_{\mathscr{C}}(C,D) & \longrightarrow & \mathrm{Hom}_{\mathscr{C}}(A,C) \times \mathrm{Hom}_{\mathscr{C}}(C,D) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathscr{C}}(A,B) \times \mathrm{Hom}_{\mathscr{C}}(B,D) & \longrightarrow & \mathrm{Hom}_{\mathscr{C}}(A,D) \end{array}$$

commutes for every A, B, C and D, and for every X there is some $1_X \in \mathrm{Hom}_{\mathscr{C}}(X,X)_0$ where $1_X \circ f = f, g \circ 1_X = g$ for all appropriate f, g.

Such a construct possibly returns to our original concept of what higher categories should be, with higher morphisms existing in the space of morphisms between two objects. Every category $\mathscr{C}$ can be seen immediately as a simplicial category by interpreting every hom-set as a discrete simplicial set. Functors between simplicial sets may now also be defined, where $F : \mathscr{C} \to \mathscr{D}$ maps objects as normal and defines maps of simplicial sets $\mathrm{Hom}(A,B) \to \mathrm{Hom}(F(A), F(B))$ respecting composition and identities as usual. This grants us a natural category **S-Cat** of small simplicial categories. We thus have a functor $R : \mathbf{Cat} \to \mathbf{S\text{-}Cat}$ sending categories to their discrete simplicial analogues.

An example of such a simplicial category is **sSet** itself, owing to the function complexes we defined. Composition here is defined level-wise as

$$\begin{aligned} &\circ_n : \mathbf{Map}(A,B)_n \times \mathbf{Map}(B,C)_n \to \mathbf{Map}(A,C)_n \\ &(f : A \times \Delta^n \to B, g : B \times \Delta^n \to C) \mapsto (g \circ (f, \mathrm{pr}_{\Delta^n}) : A \times \Delta^n \to C) \end{aligned}$$

where pr_{Δ^n} is projection onto the Δ^n component. This should correspond to our intuition about composing homotopies. Indeed, if we imagine composing standard topological homotopies $H : A \times I \to B$ and $K : B \times I \to C$, the former from f to f' and the latter from g to g', we should expect the homotopy $K \circ (H, \mathrm{pr}_I) : A \times I \to C$, as this is a homotopy from $g \circ f$ to $g' \circ f'$. To see associativity, for any $f \in \mathbf{Map}(A,B)_n$, $g \in \mathbf{Map}(B,C)_n$ and $h \in \mathbf{Map}(C,D)_n$, note that $h \circ (g, \mathrm{pr}_{\Delta^n}) \circ (f, \mathrm{pr}_{\Delta^n})$ is indeed equal to $h \circ (g \circ (f, \mathrm{pr}_{\Delta^n}), \mathrm{pr}_{\Delta^n})$. Moreover, this category is *internal* because of these same function complexes.

Another example is, in fact, the category of small categories; after all, a natural transformation of functors $F, G : \mathscr{C} \to \mathscr{D}$ is just a functor $H : \mathscr{C} \times [\overline{1}] \to \mathscr{D}$. We can extend this to functors $H : \mathscr{C} \times [\overline{n}] \to \mathscr{D}$ representing chains of n natural transformations and their compositions, inducing a simplicial structure on $\mathbf{Fun}(\mathscr{C}, \mathscr{D})$ that works for reasons similar to the simplicial set case. In reality, this is just $\mathbf{Map}(N\mathscr{C}, N\mathscr{D})$.

A rather straightforward construction available to us now is that of *truncation*, the simplicial analogue to the homotopy category:

Definition 3.26. [52, pg. 70] Given a simplicial category $\mathscr{C}$, its *truncation* $\tau_{\leq 1}\mathscr{C}$ is the category with objects $\mathrm{Ob}(\mathscr{C})$ and morphisms

$$\mathrm{Hom}_{\tau_{\leq 1}\mathscr{C}}(A, B) = \pi_0 \mathrm{Hom}_{\mathscr{C}}(A, B)$$

for every A, B.

If we interpret $\tau_{\leq 1}\mathscr{C}$ as a simplicial category itself, then we have a clear mapping $\gamma : \mathscr{C} \to \tau_{\leq 1}\mathscr{C}$ that leaves objects unchanged and acts as the quotient map $x \mapsto [x]_{\cong}$ on hom-sets. Note that $\pi_0 X \times \pi_0 Y \cong \pi_0(X \times Y)$ as sets [48, pg. 21], so this is a functor. Furthermore, we in fact have a functor $\tau_{\leq 1} : \mathbf{S\text{-}Cat} \to \mathbf{Cat}$, as functors between simplicial categories may be reduced to standard ones between their truncations by taking quotients of maps between hom-sets. This of course works in the large case as well.

How should we define full faithfulness and essential surjectivity? We would like each to be 'up to homotopy' in some sense, a concept that is best resolved by our model structure on simplicial sets:

Definition 3.27. [60, pg. 10] Let $F : \mathscr{C} \to \mathscr{D}$ be a functor between simplicial categories.

1. F is *essentially surjective* if the induced functor $\tau_{\leq 1}F : \tau_{\leq 1}\mathscr{C} \to \tau_{\leq 1}\mathscr{D}$ is as such.

2. The *essential image* of F is the preimage under the projection $\mathscr{C} \to \tau_{\leq 1}\mathscr{C}$ of $\tau_{\leq 1}F$'s essential image.

3. F is *fully faithful* if it induces weak equivalences of simplicial sets $\mathrm{Hom}_{\mathscr{C}}(A, B) \to \mathrm{Hom}_{\mathscr{D}}(F(A), F(B))$ for every A, B.

4. F is an *equivalence* if it is fully faithful and essentially surjective.

It is in fact shown in [13, pg. 10] that there is a model category structure on **S-Cat** whose weak equivalences are these equivalences. While this is a crucial result, we lack the time to prove it here.

A final construction we will consider is that of the underlying category for any simplicial category:

Definition 3.28. [52, pg. 70] The *underlying category* of a simplicial category $\mathscr{C}$ is the category $\overline{\mathscr{C}}$ whose objects are the same as $\mathscr{C}$ and morphisms are $\mathrm{Hom}_{\overline{\mathscr{C}}}(A,B) = \mathrm{Hom}_{\mathscr{C}}(A,B)_0$ for all A, B.

This object is fundamentally different from $\tau_{\leq 1}\mathscr{C}$, as the underlying category does not identify vertices while the truncation does. The forgetful functor $J : \mathbf{S\text{-}Cat} \to \mathbf{Cat}$ may be defined to restrict functors to vertices of simplicial hom-sets. As noted in [60, pg. 10], the functors we have developed thus far induce two adjunctions $R : \mathbf{Cat} \Leftrightarrow \mathbf{S\text{-}Cat} : J$ and $\tau_{\leq 1} : \mathbf{S\text{-}Cat} \Leftrightarrow \mathbf{Cat} : R$, forming a so-called *adjoint triple* $\tau_{\leq 1} \dashv R \dashv J$.

3.4.4 Defining Simplicial Localization

With our new formulation of a higher category, we may now try to construct the simplicial localization we flaunted earlier. Our work here begins with a pair $(\mathscr{C}, W)$ of a category $\mathscr{C}$ and a subcategory W. We have usually called this a class of morphisms, but demanding a subcategory is little more than a humble request for closure under composition and identities. Our goal will be to construct a simplicial category $L(\mathscr{C}, W)$ such that $\tau_{\leq 1}L(\mathscr{C}, W) \cong \mathscr{C}[W^{-1}]$. There are two ways to achieve this localization, which we will discuss here.

3.4.4.1 Standard Simplicial Localization

The first is the original standard simplicial localization, introduced by Dwyer and Kan in [13]. This method requires our category $\mathscr{C}$ to be small, yet it is still instructive to consider.

A small category is called *free* if there is some set of non-identity morphisms that freely generate every other morphism by finite compositions [13, pg. 3]. Given a small category $\mathscr{C}$ with a subcategory of weak equivalences W, we write $F\mathscr{C}$ for the free category with the same objects as $\mathscr{C}$ but with morphisms freely generated by $\mathrm{Mor}(\mathscr{C})$, the old composition operation discarded. These new generators will be called Fc for each $c \in \mathrm{Mor}(\mathscr{C})$. There is, of course, a clear functor $\phi : F\mathscr{C} \to \mathscr{C}$ doing nothing to objects but sending $Fc \mapsto c$ for each Fc. Furthermore, another functor $\psi : F\mathscr{C} \to F^2\mathscr{C}$ is defined by sending morphisms $Fc \mapsto F(Fc)$.

We have a functor $F : \mathbf{Cat} \to \mathbf{Cat}$ sending $\mathscr{C}$ to $F\mathscr{C}$ and functors g to Fg, where $Fg(Fc) = F(g(c))$ on all morphisms c. With this in mind, the functors ϕ and ψ may be seen as natural transformations $F \to 1_{\mathbf{Cat}}$ and $F \to F^2$ respectively. We may now see that F, ϕ and ψ satisfy so-called *comonad identities* [13, pg. 4]

$$\phi(F\phi) = \phi(\phi F)$$
$$(F\psi)\psi = (\psi F)\psi$$
$$(F\phi)\psi = 1_{\mathbf{Cat}} = (\phi F)\psi$$

allowing us to construct the following:

Definition 3.29. [13, pg. 4] The *standard simplicial resolution* of a small category $\mathscr{C}$, written $F_*\mathscr{C}$, is the functor $\Delta^{op} \to \mathbf{Cat}$ of the form $F_k\mathscr{C} := (F_*\mathscr{C})(\overline{n}) = F^{n+1}\mathscr{C}$ and with pullback morphisms

$$(d^i)^* = (F^i\phi F^{n-i})_{\mathscr{C}} : F^{n+1}\mathscr{C} \to F^n\mathscr{C}$$
$$(s^i)^* = (F^i\psi F^{n-i})_{\mathscr{C}} : F^{n+1}\mathscr{C} \to F^{n+2}\mathscr{C}$$

We should try to understand exactly what this construction is. Before all else, we can redefine this as a simplicial category since the objects in all the categories $F_n\mathscr{C}$ are constant and preserved identically by all pullbacks. Hence, we may redefine $F_*\mathscr{C}$ by setting its objects to be $\mathrm{Ob}(\mathscr{C})$ and setting the simplicial set

$$\mathrm{Hom}_{F_*(\mathscr{C})}(A, B)_n = \mathrm{Hom}_{F_n(\mathscr{C})}(A, B)$$

with pullback morphisms defined by the pullback functors acting on hom-sets in the original definition. These are of course simplicial sets in possession of an associative composition operation, derived from the original definition as well. Hence, we have a simplicial category. This can of course be done with any functor $\Delta^{op} \to \mathbf{Cat}$ where all categories have the same objects, preserved by pullback morphisms.

With this in mind, we should try and understand the content of this simplicial category. For any $A, B \in \mathscr{C}$, $\mathrm{Hom}_{F_*(\mathscr{C})}(A, B)_0$ is the set of paths of non-identity morphisms in $\mathscr{C}$ from A to B:

$$A \to \cdots \to B$$

Similarly, $\mathrm{Hom}_{F_*(\mathscr{C})}(A, B)_1$ is the set of paths in $F\mathscr{C}$ from A to B, ie. paths of paths in $\mathscr{C}$ overall starting at A and ending at B:

$$A(\to \cdots \to)(\to \cdots \to)\dots(\to \cdots \to)B$$

We may continue this process inductively, seeing $\mathrm{Hom}_{F_*(\mathscr{C})}(A, B)_n$ as paths nested n levels deep, starting at A and ending at B.

Consider the face operators $d^0, d^1 : \overline{0} \to \overline{1}$. In the simplicial set $\mathrm{Hom}_{F_*(\mathscr{C})}(A, B)$, these become $(\phi F)_{\mathscr{C}}$ and $(F\phi)_{\mathscr{C}}$, respectively. The first of these sends a path of paths to the single path gained by concatenation. The second applies F to the transformation sending a path to its composition in $\mathscr{C}$, so sends a path of paths to the path gained by composing each sub-path in $\mathscr{C}$.

Similarly, for $\mathrm{Hom}_{F_*(\mathscr{C})}(A, B)_2$, the elements will be morphisms in $F^3\mathscr{C}$, ie. paths of paths of paths starting at A and ending at B overall. The face operator $(d^0)^* = (\phi F^2)_{\mathscr{C}}$ will compose the overall path of morphisms in $F^2\mathscr{C}$, concatenating it into one path of paths. $(d^1)^* = (F\phi F)_{\mathscr{C}}$ will operate within each morphism in $F^2\mathscr{C}$ in the path, performing composition (concatenation) at this level. Finally, $(d^2)^* = (F^2\phi)_{\mathscr{C}}$ will go down two levels of nesting to the paths in $F\mathscr{C}$, composing them by composition in $\mathscr{C}$. All three operations reduce a morphism in $F^3\mathscr{C}$ to one in $F^2\mathscr{C}$. Similarly, the face operator d^i in dimension n will, in general, send a path of nested paths n levels deep to one $n-1$ levels deep by doing composition at the level i of nesting, where level 0 is the coarsest and n the finest, namely in $\mathscr{C}$ itself.

The degeneracy operators $(s^i)^*$ will then convert a path nested n levels deep into one $n+1$ levels, where the i^{th} level of nesting is a trivial singleton. This shows us how the standard simplicial resolution encodes nestings of paths in $\mathscr{C}$ and how they can be concatenated or composed.

It is rather clear in this case that $\tau_{\leq 1}F_*\mathscr{C} \cong \mathscr{C}$; any path of morphisms f is a face for the corresponding 1-element path of paths, whose other face is the composition of f in $\mathscr{C}$. Similarly, its underlying category is $F\mathscr{C}$. This suggests an interesting result:

Proposition 3.22. [13, pg. 4] If a small category $\mathscr{C}$ is regarded as a simplicial category, then $\phi : F_*\mathscr{C} \to \mathscr{C}$ is a weak equivalence.

Proof. The map $f \mapsto Ff$ of morphisms $f : A \to B$ in $\mathscr{C}$ yields a contracting homotopy inverse for the geometric realization of ϕ. □

With this in mind, we will perform an interesting operation on $F_*\mathscr{C}$ by, when considering it as a functor $\Delta^{op} \to \mathbf{Cat}$, *localizing* every single $F_n\mathscr{C}$ by F_nW. This will of course yield a new simplicial category.

Definition 3.30. [13, pg. 7] The *standard simplicial localization* $L(\mathscr{C}, W)$ of a category $\mathscr{C}$ with subcategory W is the simplicial category corresponding to the functor $G : \Delta^{op} \to \mathbf{Cat}$ where $G(\overline{n}) = F_n\mathscr{C}[F_nW^{-1}]$ and morphisms are localizations of the corresponding maps in the simplicial resolution.

In $G(\overline{n})$, those paths nested of nesting depth n containing only morphisms in W have been given a formal inverse. Therefore, in the simplicial category $L(\mathscr{C}, W)$, we must have that the composition of such a path and its inverse maps to a path of nesting n and length 0, ie. the identity. Formally, this is simply the n-simplex obtained by applying degenerate maps $(s^i)^*$ to the identity, since it is a 0-simplex in this context.

How does this present the homotopies we desired? As an example, consider a weak equivalence $g : A \to B$ and morphism $f : C \to A$. In the simplicial localization, let $g' : B \to A$ be the formal inverse of g. Now, the path $C \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{g'} A$ may be seen as a 1-simplex, ie. a path of length 2

$$C(\xrightarrow{f}\xrightarrow{g})(\xrightarrow{g'})B$$

in $F\mathscr{C}[FW^{-1}]$. The faces of this simplex will be the composition $C \xrightarrow{g\circ f} B \xrightarrow{g'} A$ and $C \xrightarrow{f} A$, showing how inverses of weak equivalences may act on one already composed with something else. In a normal localization, this is just done by setting both to be inverse in the usual categorical sense, while here, it is explicitly shown in a higher morphism.

We now can rather readily see the result we sought:

Proposition 3.23. [13, pg. 7] For any small category $\mathscr{C}$ with subcategory W, we have that

$$\tau_{\leq 1} L(\mathscr{C}, W) \cong \mathscr{C}[W^{-1}]$$

Another interesting property follows, which we do not prove here:

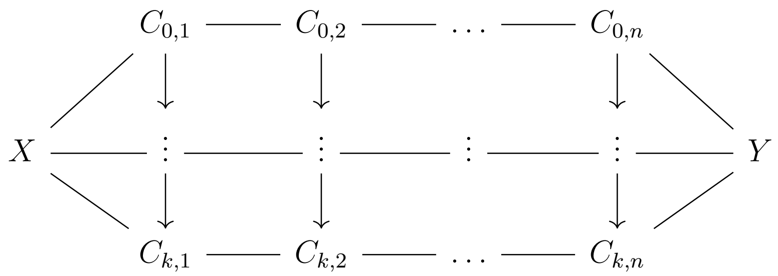
Proposition 3.24. [13, pg. 7] For a small category $\mathscr{C}$ and morphism $f : A \to B$, we have that $f \in W$ if and only if precomposition and postcomposition induce weak equivalences $\mathrm{Hom}_{L(\mathscr{C},W)}(B, C) \cong \mathrm{Hom}_{L(\mathscr{C},W)}(A, C)$ and $\mathrm{Hom}_{L(\mathscr{C},W)}(C, A) \cong \mathrm{Hom}_{L(\mathscr{C},W)}(C, B)$ for all $C \in \mathscr{C}$.

Note that these are not weak equivalences of simplicial sets if f is an isomorphism outside of W [13, pg. 7]. This is not really a problem, since we can usually assume W contains all isomorphisms as well.

3.4.4.2 Hammock Simplicial Localization

While the above construction gives some insight into our methods, we would rather take localizations in the context of large categories as well as small ones. Furthermore, as noted by Dwyer and Kan in [11, pg. 1], it is often difficult to know the homotopy types of the simplicial hom-sets of such a localization. They present a new form of localization that is equivalent to the prior one in small cases, called the *hammock localization*. We will present it here and use it henceforth.

Definition 3.31. [11, pg. 3] The *hammock localization* of a category $\mathscr{C}$ with subcategory W, written $L^H(\mathscr{C}, W)$, is the simplicial category sharing objects with $\mathscr{C}$ but where $\mathrm{Hom}_{L^H(\mathscr{C},W)}(A, B)$ is the simplicial set whose k-simplices are 'hammocks with k rows and any number of columns', ie. diagrams



such that:

1. $n \geq 0$ is an integer.
2. All vertical maps are in W.
3. All maps in a column go in the same direction; if they go left, they are in W.
4. Maps in adjacent columns go in different directions.
5. No column contains only identity maps.

The i^{th} face map removes the i^{th} row and the i^{th} degeneracy repeats it. Composition is done by concatenating hammocks left-to-right and using compositions to produce a new hammock. If any of these operations invalidates properties 4 or 5 above, we then aggressively compose columns facing the same direction and omit those containing only identity maps.

This is a far more natural definition than the standard simplicial localization, if one thinks hard enough - a 0-simplex in a simplicial hom-set now describes a path $A \longleftrightarrow \cdots \longleftrightarrow B$ where leftward arrows are weak equivalences. The strategy here basically treats weak equivalences as though they have inverses, meaning there is no issue with having them in the middle of a path in the wrong direction. In some sense, such a path is a path through 'objects up to W'. A more general n-simplex is just like the n-simplices in a nerve category now, namely representing a path through

0-simplices. In this case, the edges of the n-simplex in question must consist only of weak equivalences, directly showing how two paths are the same up to W.

To form this into a rather enlightening example, consider a morphism $f : A \to B$ in W. We have the 1-simplex in $L^H(\mathscr{C}, W)$

$$\begin{array}{ccccc} A & \xrightarrow{1_A} & A & \xleftarrow{1_A} & A \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle f} & \swarrow{\scriptstyle f} & \\ & & B & & \end{array}$$

In this case, it is clear that the 0-simplex $B \xleftarrow{f} A$ is a right inverse to f in the homotopy category $\tau_{\leq 1} L^H(\mathscr{C}, W)$, as a path of identities will be reduced to a single identity by our definition. Clearly it will be a left inverse as well, so in fact f is invertible in the truncated category. This suggests the following result, proved in a more natural way we won't explore in [11]:

Proposition 3.25. [11, pg. 4] $\tau_{\leq 1} L^H(\mathscr{C}, W)$ is equivalent to $\mathscr{C}[W^{-1}]$ for any category $\mathscr{C}$ and subcategory W.

Some other natural results present themselves, such as $L^H(\mathscr{C}, W)$ being weakly equivalent to $L(\mathscr{C}, W)$ in the small case [11, pg. 3], which we abstain from proving here. What this does mean is that we may consider hammock localizations in place of standard ones without losing any homotopical data.

While localizations seem to be a particular case of simplicial category, it turns out they are in fact exceedingly common:

Proposition 3.26. [43] Up to equivalence of simplicial categories, every simplicial category is the simplicial localization of a category $\mathscr{C}$ with subcategory W containing all isomorphisms and satisfying the 2-out-of-3 property on morphisms.

While we will not prove this fact here, the structure it describes seems rather close to that of a model category, which we turn to next.

3.4.4.3 Localizing Model Categories

Perhaps the most interesting case of localization is when we further demand our category $\mathscr{C}$ to be a *model category*, with W its weak equivalences. In such a scenario, we will write $L\mathscr{M}$ or $L^H\mathscr{M}$ for the localization or hammock localization along the weak equivalences, respectively, though we will often use these somewhat interchangeably.

We would hope that doing this would free us of the constraints of fibrant and cofibrant objects, as well as making all higher homotopical data accessible in a way that the homotopy category does not.

To begin, the desire for $L\mathscr{M}$ to be independent of fibrations and cofibrations is immediately granted by definition. The only real concern to address is that losing this information may then bar us access to that homotopical information we care for in $\mathscr{M}$. Clearly at the level of path components, this is sustained, though this is not quite all we were looking for; if it were, normal localization would be sufficient.

How are we to define higher homotopies in a model category? Consider a cofibrant object A and fibrant B. Having seen the cylinder and path objects $A \wedge I$ and B^I, we will informally replace I with Δ^1 and generalize these objects to higher $A \wedge \Delta^n$ and B^{Δ^n} to better hone our intuition.

Departing somewhat from the existing literature, we turn our attention to the diagram

$$\cdots \; \overrightarrow{\overrightarrow{\overrightarrow{\rightarrow}}} \; B \times B \times B \; \overrightarrow{\overrightarrow{\rightarrow}} \; B \times B \; \overrightarrow{\rightarrow} \; B$$

where morphisms at each level are projections $\mathrm{Pr}_i : B^n \to B^{n-1}$ excluding one term i in the product[3]. There is another diagram of the exact same form as the one above but with all arrows reversed, where each morphism $D_i : B^n \to B^{n+1}$ duplicates the i^{th} and $(i+1)^{th}$ values for some i, ie. a generalized diagonal.

With these in mind, we can form a functor $\Delta^{op} \to \mathscr{M}$ sending $\overline{n}$ to B^{n+1} and face and degeneracy operators to the projection and diagonal morphisms we outlined above. In general, a functor $\Delta^{op} \to \mathscr{M}$ is called a *simplicial object*. A simplicial set is then just a simplicial object in the category of sets.

We now define a higher path object B^{Δ^n} by a weak factorization of the diagonal $B \to B^n$

$$B \xrightarrow{w_n} B^{\Delta^n} \xrightarrow{v_n} B \times \cdots \times B$$

where w_n is an acyclic cofibration and v_n a fibration. We get morphisms $p_i^n : B^{\Delta^n} \to B$ in much the same manner as for B^I, which we can assume to be acyclic fibrations as B is fibrant and by a straightforward extension of the proof for Proposition 3.8. We will also write $B^{\Delta^0} = B$ and $w_0 = v_0 = 1_B$, which clearly satisfy the above requirements.

We would like a simplicial object valued in higher path objects rather than products. To declare such an object, for any $f : B^{n+1} \to B^{m+1}$ in the image of our

[3]These should not be confused with the projections $\mathrm{pr}_i : B^n \to B$.

simplicial object, we can take lifts of the form

$$\begin{tikzcd} B \arrow[rr, "w_m"] \arrow[d, "w_n"'] & & B^{\Delta^m} \arrow[d, "v_m"] \\ B^{\Delta^n} \arrow[r, "v_n"] \arrow[urr] & B^{n+1} \arrow[r, "f"] & B^{m+1} \end{tikzcd}$$

as our operators. Notably, this choice can be made such that it respects composition, as for morphisms $f : B^{n+1} \to B^{m+1}$ and $g : B^{m+1} \to B^{p+1}$ with lifts f' and g' respectively, the diagram

$$\begin{tikzcd} & B \arrow[dl, "w_n"'] \arrow[d, "w_m"] \arrow[dr, "w_p"] & \\ B^{\Delta^n} \arrow[r, "f'"] \arrow[d, "v_n"'] \arrow[rr, bend right] & B^{\Delta^m} \arrow[r, "g'"] \arrow[d, "v_m"] & B^{\Delta^p} \arrow[d, "v_p"] \\ B^{n+1} \arrow[r, "f"] & B^{m+1} \arrow[r, "g"] & B^{p+1} \end{tikzcd}$$

indicates how the composition $g' \circ f'$ is always a viable lift of $g \circ f$. Hence, we could start with lifts of face and degeneracy operators and then take iterative compositions of these. We should also note that all our chosen morphisms are weak equivalences by 2-out-of-3.

We could now assemble a (not entirely commutative) diagram of the form

$$\begin{tikzcd} B \arrow[r, leftrightarrow] & B \times B \arrow[r, leftrightarrow] & B \times B \times B \arrow[r, leftrightarrow] & \cdots \\ B \arrow[r, leftrightarrow] \arrow[u, "1_B"] & B^{\Delta^1} \arrow[r, leftrightarrow] \arrow[u, "v_1"] & B^{\Delta^2} \arrow[r, leftrightarrow] \arrow[u, "v_2"] & \cdots \end{tikzcd}$$

where we have written $\leftrightarrow$ as a placeholder for all the face and degeneracy morphisms we chose. All the morphisms in the top row exactly correspond to ones in the bottom row such that composition is preserved, so we can follow this correspondence and produce a new simplicial object $P : \Delta^{op} \to \mathscr{M}$ sending $\overline{n} \mapsto B^{\Delta^n}$.

What has this construction gained us? Consider morphisms into elements of this simplicial object. A morphism $A \to B$ is a usual morphism and a morphism $A \to B^{\Delta^1}$ is a right homotopy. For $A \to B^{\Delta^2}$, the face operators we chose let us indicate three right homotopies related by this map, meaning this morphism is one between homotopies. As we saw with simplicial sets, the morphisms $A \to B^{\Delta^n}$ now indicate all the 'higher right homotopies' we should naturally expect.

A dual construction for cofibrant A gives us a functor $\Delta \to \mathscr{M}$, or a *cosimplicial object*, mapping $\overline{n} \mapsto A \wedge \Delta^n$, where the *higher cylinder object* $A \wedge \Delta^n$ is constructed dually to higher path objects. A similar analysis shows how morphisms $A \wedge \Delta^n \to B$ indicate higher left homotopies.

It would be desirable if these higher path objects and morphisms could be chosen such that all face operations are fibrations; intuitively, it should be possible to lift homotopies up to higher ones. While all our higher path objects will be at least fibrant owing to the morphism p_i^n being a fibration, this tragically does not seem obvious in our current setup, so we leave a direct proof as future work. A dual expectation may then be had of higher cylinder objects.

Fortunately, all is not lost: Dwyer and Kan in [12, pg. 5-6] define a construction that essentially generalizes this intuition to discuss higher morphisms of objects in a model category, enforcing our wish for all face operations to indeed be fibrations:

Definition 3.32. [12, pg. 5] A *simplicial resolution* of an object X in a model category $\mathscr{M}$ is a simplicial object $X_* : \Delta^{op} \to \mathscr{M}$ with an acyclic fibration $X \to X_0$ such that:

1. X_0 is fibrant.
2. All face maps are acyclic fibrations (meaning all X_n are fibrant).
3. For every $n \geq 0$ the induced map $X_{n+1} \to (d_*, X_n)$ is a fibration, where (d_*, X_n) is the limit of the diagram consisting of
   (a) a copy (d_i, X_n) of X_n for every $i \in \overline{n+1}$ and
   (b) a copy (d_{ij}, X_{n-1}) of X_{n-1} for every $0 \leq i < j \leq n+1$ and a cospan $(d_j, X_n) \overset{(d^{j-1})^*}{\to} (d_{ij}, X_{n-1}) \overset{(d^i)^*}{\leftarrow} (d_i, X_n)$.

An entirely dual construction to this is a *cosimplicial resolution.*

A morphism between (co)simplicial resolutions of a fixed object are natural transformations respecting the defining (co)fibrations.

The last point here basically stipulates that, if we are willing to imagine a complex of the form $\partial\Delta^n$ in our resolution by higher path objects, the pullback of the inclusion $\partial\Delta^n \to \Delta^n$ should be a fibration. This is also a perfectly natural property to expect, as lifting along such a map should be feasible.

Such an abstract construction is one that is thankfully always present:

Proposition 3.27. [12, pg. 6] Every object in a model category has a simplicial and cosimplicial resolution.

We may now, given a simplicial resolution B_* of some B and a cofibrant A, construct a simplicial set $\text{Hom}(A, B_*)$ whose sets of simplices are $\text{Hom}(A, B_n)$ and pullback morphisms are defined by postcomposition. We can do a dual construction for a cosimplicial resolution A^* and fibrant B to get a simplicial set $\text{Hom}(A^*, B)$. These should both contain all the higher homotopical information between A and B we sought before.

With this construction available to us, we can now state the major result:

Proposition 3.28. [12, pg. 6] Let $\mathscr{M}$ be a model category with objects X and Y, such that X has a cosimplicial resolution X^* and Y a simplicial resolution Y_*. If X is cofibrant, then the simplicial set $\text{Hom}(X, Y_*)$ is weakly equivalent to $\text{Hom}_{L^H\mathscr{M}}(A, B)$. If Y is fibrant, then this is the case for $\text{Hom}(X^*, Y)$.

It is interesting that this is independent of the choice of simplicial resolution, justifying the abstraction we made in defining them. However, this still seems somewhat contrived; why are we restraining ourselves to fibrant and cofibrant objects? We should be rid of these technical obstructions in our important declarations.

Fortunately, all is not lost. To cement our intuition, we will return briefly to the case of path objects and cylinder objects. Consider two arbitrary objects A' and B', together with a cofibrant replacement A and fibrant replacement B which have a resolution by higher cylinder objects A^* and a resolution by higher path objects B_* respectively. We may consider what kinds of vertical morphisms may exist in the (not necessarily commutative) diagram

$$\begin{array}{ccccccccc} A' & \longleftarrow & A & \longleftrightarrow & A \wedge \Delta^1 & \longleftrightarrow & A \wedge \Delta^2 & \longleftrightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ B' & \longrightarrow & B & \longleftrightarrow & B^{\Delta^1} & \longleftrightarrow & B^{\Delta^2} & \longleftrightarrow & \cdots \end{array}$$

Of course, starting from the left, morphisms in the first two columns are just morphisms $A' \to B'$. In the next column however, we have morphisms $A \wedge \Delta^1 \to B^{\Delta^1}$. Note that these are, in fact, just a rather convoluted way of representing homotopies. Following face operators $B^{\Delta^1} \to B$ would produce left homotopies and following codegeneracy operators $A \wedge \Delta^1 \to A$ would do the same with right homotopies. Inspired by this, we should expect the same of general cosimplicial and simplicial resolutions.

We can create a functor $\Delta^{op} \times \Delta^{op} \to \mathbf{Set}$ sending $(\overline{n}, \overline{m}) \mapsto \text{Hom}(A^n, B_m)$ and simplicial operators to precomposition and postcomposition. The diagonal functor $\Delta^{op} \to \Delta^{op} \times \Delta^{op}$ lets us make this into a simplicial set $\text{diag}(\text{Hom}(A^*, B_*))$.

Proposition 3.29. [12, pg. 6] Let $\mathscr{M}$ be a model category with objects X and Y, such that X has a cosimplicial resolution X^* and Y a simplicial resolution Y_*. Then the simplicial set $\mathrm{diag}(\mathrm{Hom}(A^*, B_*))$ is weakly equivalent to $\mathrm{Hom}_{L^H\mathscr{M}}(A, B)$.

Unfortunately, we lack the space to prove this result here. However, this should cement the significance of hammock localizations as a tractable construction that represents higher homotopies, naturally extending the homotopy category. This lets us construct objects like the *simplicial monoid* $\mathrm{Hom}_{L^H\mathscr{M}}X$ of homotopy self-equivalences for any $X \in \mathscr{M}$, as well as the *homotopy automorphism complex* $\mathrm{haut}_{L^H\mathscr{M}}X$, the subset of the simplicial monoid containing only components that become invertible in the homotopy category [12, pg. 6]. It turns out that these are in general weakly equivalent [12, pg. 6], so either is acceptable.

A useful fact about these localizations is that they explicitly encode the nice behavior of fibrant and cofibrant objects, a fact made clear by the following:

Proposition 3.30. [12, pg. 7] Let $\mathscr{M}$ be a model category, with subcategories $\mathscr{M}^f$ of fibrant objects and $\mathscr{M}^c$ of cofibrant ones, with induced model structures. Then the induced inclusions

$$L^H(\mathscr{M}^c, W^c \cap \mathrm{Cofib}) \to L^H(\mathscr{M}^c, W^c) = L^H\mathscr{M}^c \to L^H\mathscr{M}$$
$$L^H(\mathscr{M}^f, W^f \cap \mathrm{Fib}) \to L^H(\mathscr{M}^f, W^f) = L^H\mathscr{M}^f \to L^H\mathscr{M}$$

are all weak equivalences.

A full proof is given in [12].

Another appealing fact about localizations is that, much like homotopy categories, they behave well under Quillen pairs. In some sense, they extend the behavior of homotopy categories in such a scenario:

Proposition 3.31. [12, pg. 7] Let $S : \mathscr{M} \Leftrightarrow \mathscr{N} : T$ be a Quillen pair. Then, for every cofibrant X with cosimplicial resolution X^* and fibrant Y with a simplicial[4] resolution Y_*, the adjunction map induces a natural isomorphism

$$\mathrm{Hom}(X^*, TY_*) \cong \mathrm{Hom}(SX^*, Y_*)$$

Moreover, if for every cofibrant $X \in \mathscr{M}$ and fibrant $Y \in \mathscr{N}$, a map $X \to TY$ is a weak equivalence if and only if its adjoint $SX \to Y$ is so, then the induced functors $L^H\mathscr{M}^c \to L^H\mathscr{N}^c$ and $L^H\mathscr{N}^f \to L^H\mathscr{M}^f$ are equivalences of simplicial categories.

[4]This says cosimplicial in the original proposition and omits 'fibrant', which I assume to be typos.

3.5 Segal Categories

Before we conclude our expedition through the realm of higher category theory, it will be in our best interest to weaken our notion of a simplicial category even further than we have done. Indeed, if we were to leap straight into defining higher stacks, we would soon trip over our own shoelaces as we tried to grapple with any internal structure simplicial category theory has to offer.

There is a natural notion of a simplicial category of functors $\mathbf{Map}(\mathscr{C}, \mathscr{D})$ between simplicial categories $\mathscr{C}$ and $\mathscr{D}$, whose simplicial sets of morphisms come from natural transformations along simplicial hom-sets in the source and target categories [26, pg. 70]. Unfortunately, this turns out not to be invariant under equivalences of simplicial categories [58, pg. 14], so we are somewhat at a loss for defining a higher category of functors 'up to homotopy'. A more precise discussion of this phenomenon may be found in the remark in [26, pg. 70-71].

It turns out that loosening the currently tight collar of our composition operation will not only solve this problem, but also yield a number of useful properties, not the least of which is a rather fascinating connection to model categories called the *strictification theorem*.

What is it exactly that we want to achieve? Consider a nerve $N(\mathscr{C})$. If we were to try and understand composition in this context, we would wind up with a picture of the form

$$\begin{array}{ccc} N(\mathscr{C})_2 & \xrightarrow{(\langle 0,1\rangle^*, \langle 1,2\rangle^*)} & N(\mathscr{C})_1 \times_{N(\mathscr{C})_0} N(\mathscr{C})_1 \\ {\scriptstyle \langle 0,2\rangle^*}\downarrow & & \\ N(\mathscr{C})_1 & & \end{array}$$

where the fibre product $N(\mathscr{C})_1 \times_{N(\mathscr{C})_0} N(\mathscr{C})_1$ is with respect to the target and source maps. Hence, the horizontal map is, in this context, a bijection. This bijection yields our notion of composition in a nerve category. However, for a general simplicial set, there really are no requirements on this horizontal map at all.

We would like to see the horizontal map as an equivalence 'up to homotopy'. One way to achieve this is through a special kind of simplicial set where the horizontal map is a surjection, called a *quasicategory* [48]. This notion has been developed extensively by Joyal and Lurie [52, pg. 81]. However, we will not employ it here, as the correspondence with simplicial categories is not so clear. Instead, we will replace all the sets in the above diagram with *simplicial sets*, demanding that a new version of the horizontal arrow above is a weak equivalence. It will turn out that simplicial categories can be seen this way as well, where the horizontal map is an isomorphism.

3.5.1 Segal's Delooping Machine

The story of Segal categories begins in algebraic topology, as explained by Simpson in [52, pg. 71]. We will follow his exposition here. For a pointed topological space (X, x), consider the corresponding *loop space* ΩX of all continuous maps $(S^1, 1) \to (X, x)$. A composition operation may be defined just as with the fundamental group, which is of course simply a quotient of ΩX by certain homotopies. Hence, it is often possible to replace this loop space with certain topological groups, from which one may produce a 'classifying space' construction $B(-)$ such that $B(\Omega X) \sim X$. The question was whether one could produce a form of space homotopy equivalent to ΩX with weaker structure than this topological group notion, yet which could yield X again via some $B(-)$ construction. We dub this operation *delooping*.

Segal proposed a *delooping machine* of the following form. Consider a simplicial set X where $X_0 = \{*\}$ is a singleton and such that the operations $(\langle 0,1\rangle^*, \ldots, \langle n-1, n\rangle^*)$ of the form

$$X_n \to X_1 \times X_1 \times \cdots \times X_1$$

which we call the *Segal maps*, are bijections. We get a model for a monoid this way; indeed, the picture

$$\begin{array}{ccc} X_2 & \xrightarrow{\cong} & X_1 \times X_1 \\ \downarrow & & \\ X_1 & & \end{array}$$

of the same form as the similar span we considered for nerves gives a notion of composition and the degeneracy map $X_0 \to X_1$ gives an identity. X_3 lets us consider associativity, as usual.

Simpson explains that Segal's insight was, in essence, to replace these simplicial sets with functors

$$X : \Delta^{op} \to \mathbf{Top}$$

where X_0 is the singleton and change the Segal maps

$$X_n \to X_1 \times X_1 \times \cdots \times X_1$$

into weak homotopy equivalences such that the monoid $\pi_0 \circ X$ is a group. The delooping of such an object is then accomplished by first taking $Y = \mathrm{Sing} \circ X$, giving us a functor $Y : \Delta^{op} \to (\Delta^{op} \to \mathbf{Set})$, which can be replaced with a *bisimplicial set* by first applying the Cartesian closed structure on **Cat** [31, pg. 98] to get an equivalent $Y : \Delta^{op} \times \Delta^{op} \to \mathbf{Set}$ and precomposing with the diagonal $\Delta_{\Delta^{op}} : \Delta^{op} \to \Delta^{op} \times \Delta^{op}$ to get the *diagonal realization* simplicial set.

The details of this construction's significance in algebraic topology are not so interesting to us; the intrigued reader is referred to [49], where Segal used somewhat different machinery to Simpson's exposition. Nevertheless, after appearing again in work of Dwyer, Kan and Smith in [14] under a different name, Tasmani in [55] took inspiration from Segal's idea and suggestions of Grothendieck and began the development of what we now know as a Segal category today [52, pg. 72].

3.5.2 Defining Segal Categories

Before we define a Segal category, we must first consider the more general *Segal precategories.*

Definition 3.33. [52, pg. 74] A *Segal precategory* is a bisimplicial set, ie. a functor $A : \Delta^{op} \times \Delta^{op} \to \mathbf{Set}$, such that the simplicial set $A(\overline{0}, -)$ is discrete. We write $A_{m,n} := A(\overline{m}, \overline{n})$.

We will write **PrSeCat** for the category of all Segal precategories, with natural transformations as morphisms, following the notation of [62, pg. 4]. We will also write $A_{p/}$ to mean the simplicial set $k \mapsto A_{p,k}$ for any $p \geq 1$, choosing to see A_0 as a set more than anything [52, pg. 74]. In this sense, each level of higher morphism gets its own independent notion of weak equivalence.

Reusing the Cartesian closed structure on **Cat**, we may interpret any Segal precategory $A : \Delta^{op} \times \Delta^{op} \to \mathbf{Set}$ as a simplicial object in **sSet** of the form $A : \Delta^{op} \to \mathbf{sSet}$. This means we get natural operators $f^* : A_{n/} \to A_{m/}$ for every $f : \overline{m} \to \overline{n}$ in Δ. Of course, when restricted to the underlying sets of objects $A_{n,0}$ this structure is exactly a simplicial set; it now simply respects the higher morphisms each level of morphism possesses.

In the end, what we are really trying to capture is the fact that Segal maps in Segal's delooping machine have the privilege of being called 'weak equivalences'. To formalize this idea, consider the fibre products $A_{1/} \times_{A_0} A_{1/}$ from the span $\langle 1 \rangle^* : A_{1/} \to A_0 \leftarrow A_{1/} : \langle 0 \rangle^*$. We will define the *Segal maps* to be those maps $(\langle 0, 1 \rangle^*, \ldots, \langle n - 1, n \rangle^*)$ taking the form

$$A_{n/} \to A_{1/} \times_{A_0} A_{1/} \times_{A_0} \cdots \times_{A_0} A_{1/}$$

[52, pg. 74].

Definition 3.34. [52, pg. 75] A *Segal category* A is a Segal precategory whose Segal maps $A_{n/} \to \ldots$ are all weak equivalences of simplicial sets, for $n \geq 2$.

Henceforth, we will write $A_{(a_0,\ldots,a_n)}$ to mean the preimage of $(a_0,\ldots,a_n) \in (A_0)^{n+1}$ under the map $(\langle 0\rangle^*,\ldots,\langle n\rangle^*) : A_{n/} \to (A_0)^{n+1}$. With this in mind, it is possible to rewrite the Segal category condition, as done in [62, pg. 6], as the requirement that for all $n \geq 2$ and $(a_0,\ldots,a_n) \in (A_0)^{n+1}$, the morphism

$$A_{(a_0,\ldots,a_n)} \to A_{(a_0,a_1)} \times A_{(a_1,a_2)} \times \cdots \times A_{(a_{n-1},a_n)}$$

is a weak equivalence of simplicial sets. Two facts become somewhat more obvious at this point. The first is that we may draw a diagram

$$\begin{array}{ccc} A_{(a,b,c)} & \longrightarrow & A_{(a,b)} \times A_{(b,c)} \\ \downarrow & & \\ A_{(a,c)} & & \end{array}$$

where the horizontal arrow is a weak equivalence of simplicial sets. This captures the weakened notion of composition that we had sought out. The other observation we now make is that any (small) simplicial category $\mathscr{C} \in$ **S-Cat** can almost immediately be turned into a Segal category C by setting

$$C_{n/} := \coprod_{(x_0,\ldots,x_n)\in \mathrm{Ob}(\mathscr{C})^{n+1}} \mathrm{Hom}_{\mathscr{C}}(x_0,x_1) \times \cdots \times \mathrm{Hom}_{\mathscr{C}}(x_{n-1},x_n)$$

[52, pg. 75]. Such objects have the nice property that all the Segal maps are in fact isomorphisms in **sSet** [62, pg. 6], so the Segal category condition is indeed satisfied. This of course gives us a way to convert any small category into a Segal category, by first seeing it as a simplicial one. Conversely, every Segal category where all Segal maps are isomorphisms clearly comes from a unique simplicial category. Hence, we may see **S-Cat** as a subcategory of **PrSeCat** with an inclusion functor I, as it is clear that this induces appropriate morphisms on Segal categories.

This inclusion functor actually takes part in a far more interesting collaboration between Segal categories and simplicial categories; Bergner established in [5] that this functor in fact has a left adjoint, which forms a Quillen equivalence with regards to certain model structures on **S-Cat** and **PrSeCat**, where the former's has equivalences of simplicial categories as its weak equvialences. We are not so interested in the model structure this uses for **PrSeCat**, as there is a far more interesting one proposed by Simpson and Hirschowitz in [26, pg. 21-22][5]. We will need some technical machinery before we can access it.

[5]It is claimed by Toën somewhat ambiguously in [57, pg. 5-6] that Bergner and Simpson's model structures are actually the same, but I fail to see the connection. It matters little for this discussion.

Definition 3.35. [5, pg. 11] For any Segal precategory A, the *homotopy category* $\mathbf{Ho}(A)$ is the category whose objects are A_0 considered as a set and morphisms are

$$\mathrm{Hom}_{\mathbf{Ho}(C)}(X,Y) := \pi_0 A_{(X,Y)}$$

for all $X, Y \in A_0$.

Definition 3.36. [26, pg. 19] A morphism $f : A \to B$ between Segal precategories is called *fully faithful* if every induced morphism $f_{(a,b)} : A_{(a,b)} \to B_{(a,b)}$ is a weak equivalence of simplicial sets and *essentially surjective* if the induced morphism $\mathbf{Ho}(f) : \mathbf{Ho}(A) \to \mathbf{Ho}(B)$ is essentially surjective. Furthermore, f is called an *equivalence* if it is both fully faithful and essentially surjective.

One may note that the functor I actually sends fully faithful and essentially surjective morphisms in the sense of simplicial categories to ones in the sense of Segal precategories. This correspondence goes further, once we observe the following fundamental fact about Segal precategories discovered by Simpson and Hirschowitz:

Proposition 3.32. [26, pg. 21-22] There exists a functor $SeCat : \mathbf{PrSeCat} \to \mathbf{PrSeCat}$ and natural transformation $i : 1_{\mathbf{PrSeCat}} \to SeCat$ such that:

1. For every $A \in \mathbf{PrSeCat}$, $SeCat(A)$ is a Segal category.
2. $i_A : A \to SeCat(A)$ is bijective on objects and an equivalence of Segal categories if A is a Segal category itself.
3. $SeCat(i_A)$ is an equivalence of Segal categories for all $A \in PrSeCat$.

Moreover, this pair $(SeCat, i)$ is unique in the sense that for any two pairs (F^1, j^1) and (F^2, j^2) satisfying the above properties, the morphism

$$F^1(j^2_A) : F^1(A) \to F^1(F^2(A))$$

is an equivalence of Segal categories for every $A \in \mathbf{PrSeCat}$ and for any $f \in \mathrm{Mor}(\mathbf{PrSeCat})$, we have that $F^1(f)$ is an equivalence precisely when $F^2(f)$ is.

We lack the time to prove this in its entirety, as it requires machinery we haven't touched on; the intrigued reader is directed to [26, pg. 22]. From this, we gain an important structure on Segal precategories:

Proposition 3.33. [26, pg. 23] **PrSeCat** admits a model structure whose weak equivalences are those morphisms f sent by $SeCat$ to an equivalence of Segal categories and whose cofibrations are all the monomorphisms.

This structure is rather endearing, as the functor $I : \mathbf{S\text{-}Cat} \to \mathbf{PrSeCat}$ sends fully faithful and essentially surjective functors in the simplicial category sense to ones in the Segal category sense. Hence, this functor preserves weak equivalences in the model structures we've alluded to.

The only finicky element of this model structure seems to be describing the fibrations. Nevertheless, we are able to gleam interesting information from the fibrant objects:

Proposition 3.34. [26, pg. 85] For every fibrant Segal category A in the above model structure, there is a simplicial category C and equivalence $C \to A$.

Indeed, this suggests, as fibrant replacements of Segal categories will be equivalent to them, that *every* Segal category is isomorphic to a simplicial one in $\mathbf{Ho}(\mathbf{PrSeCat})$. This emphasizes how Segal categories are precisely a weakening of small simplicial categories to weak equivalences.

An incredibly useful fact about the model structure we have identified is that, for any $A, B \in \mathbf{PrSeCat}$, there is a natural equivalence of Segal precategories $SeCat(A \times B) \to SeCat(A) \times SeCat(B)$, where the product is defined levelwise [62, pg. 8]. This turns out to imply that **PrSeCat** is a *monoidal model category* [62, pg. 8] [28, pg. 109], which for our purposes is just a model category whose direct product is compatible with the model structure. What matters is that this implies that $\mathbf{Ho}(\mathbf{PrSeCat})$ is Cartesian closed, as shown by Hovey in [28, pg. 115]. Hovey's proof also implies that, as **PrSeCat** is closed due to Simpson and Hirschowitz [26, pg. 112] and every object is already cofibrant, the internal hom's of $\mathbf{Ho}(\mathbf{PrSeCat})$ take the form

$$\mathbb{R}\mathrm{Hom}(A, B) := \mathbf{Hom}(A, RB)$$

where RB is a fibrant replacement for B and $\mathbf{Hom}(-,-)$ is the internal hom in **PrSeCat**. We may now see this as modelling morphisms of Segal categories up to homotopy.

The skeptical reader may notice that, if **S-Cat** does indeed have a model structure whose weak equivalences are equivalences as in [5, pg. 9], the inclusion functor I preserves equivalences and every Segal category is isomorphic to a simplicial category in $\mathbf{Ho}(\mathbf{PrSeCat})$, then we should have that $\mathbf{Ho}(\mathbf{S\text{-}Cat}) \cong \mathbf{Ho}(\mathbf{PrSeCat})$, meaning $\mathbf{Ho}(\mathbf{S\text{-}Cat})$ should be Cartesian closed itself. This is indeed the case [62, pg. 13], yet the indirection involved means we cannot explicitly define the $\mathbb{R}\mathrm{Hom}$'s in **S-Cat** in terms of internal hom's of cofibrant and fibrant replacements via [28, pg. 115]. Segal categories are evidently necessary to do this. Furthermore, this internal structure cannot possibly be equivalent to **PrSeCat**, for homotopical reasons given in [26, pg. 70-71]. Thus, Segal categories remain crucial to have an explicit internal theory consistent with homotopies.

It is not clear when exactly these precategories are Segal categories themselves, though our functor *SeCat* may let us convert them into one uniquely without changing the objects, if need be. Toën in [62] and [57] proclaims these to be Segal categories where A and B are Segal categories, so we will often take this somewhat for granted when it matters. A more satisfying proof of this property is left as future work.

One also quickly notices from the Cartesian closed structure that we have an isomorphism

$$\mathbb{R}\mathrm{Hom}(A \times B, C) \simeq \mathbb{R}\mathrm{Hom}(A, \mathbb{R}\mathrm{Hom}(B, C))$$

of precategories in $\mathbf{Ho}(\mathbf{PrSeCat})$ [62, pg. 8].

One point we should address about Segal categories in general is that they are invariably small. Such a consideration does not concern the likes of Toën and Vezzosi in [60], [61] or [62], who explicitly use a Grothendieck universe foundation for their category theory. While we won't force this constraint here, the reader should be cautious if they choose to employ Segal category theory in a context where such a foundation is not applicable.

3.5.3 Adjoints and Limits

It is a critical fact about Segal categories that they possess a straightforward concept of adjunction generalizing that of normal categories, which will grant us access to limits and colimits within them.

Definition 3.37. [62, pg. 9] [26, pg. 86] A morphism $f : A \to B$ between Segal categories is *left adjoint* to $g : B \to A$ if there is an element $u \in \mathbf{Hom}(A, A)_{(1_A,(g\circ f))}$ such that for any $a \in A_0$ and $b \in B_0$, the natural morphism between simplicial sets

$$B_{(f(a),b)} \overset{g_{(f(a),b)}}{\to} A_{(g(f(a)),g(b))} \overset{u^*}{\to} A_{(a,g(b))}$$

is a weak equivalence of simplicial sets.

It is the case that such a pair (g, u) is unique up to equivalence when it exists [62, pg. 9], so we can always speak of what we will call *the* right adjoint g of f. The transformation u, analogous to the adjunction unit in an adjoint pair for standard categories, can be replaced with a similarly unique counit $v \in \mathbf{Hom}(B, B)_{((f\circ g),1_B)}$, giving a similar condition to the one above [26, pg. 86]. A dual definition gives us right adjoints [62, pg. 9]. We then declare a morphism $f : A \to B$ in $\mathbf{Ho}(\mathbf{PrSeCat})$ to be left adjoint to another if one of its representatives $f : A' \to B'$ for fibrant A' and B' is as such, with a dual definition for right adjoints [62, pg. 9].

With this, we may now define a limit and colimit in a Segal category as in [62, pg. 9]. Let I and A be two Segal categories. The natural projection functor $A \times I \to A$, by our observations about $\mathbb{R}\mathrm{Hom}$, induces an adjoint map $c : A \to \mathbb{R}\mathrm{Hom}(I, A)$ in $\mathbf{Ho}(\mathbf{PrSeCat})$. This is the *constant diagram functor*, analogous to $\Delta : A \to \mathbf{Fun}(I, A)$ for limits and colimits in categories.

Definition 3.38. [62, pg. 9] For a constant diagram functor $c : A \to \mathbb{R}\mathrm{Hom}(I, A)$ defined as above, the *limit* and *colimit* functors

$$\lim_I : \mathbb{R}\mathrm{Hom}(I, A) \to A \qquad \mathrm{colim}_I : \mathbb{R}\mathrm{Hom}(I, A) \to A$$

are the right and left adjoints to c, respectively. The Segal category A is said to *have limits along* I if $\lim_I$ exists and *have colimits along* I if colim_I exists. If this holds for any I, then A is said to simply have limits or has colimits, respectively.

If A has limits along the categories $\emptyset$ and $\{2 \to 0 \leftarrow 1\}$, we say it has *finite limits.* If A^{op} has all finite limits, then A is said to have *finite colimits.*

The notion of an opposite Segal category A^{op} is rather straightforward; it is simply the one where $A_0^{op} = A_0$ and $A^{op}_{(a_0,\ldots,a_n)} = A_{(a_n,\ldots,a_0)}$ for all $a_0, \ldots, a_n$ with pullback morphisms redefined accordingly. We will now also use the standard notation of various kinds of limits and colimits, like products, coproducts and so on within a Segal category when we mean the limits and colimits defined above.

As noted in [62, pg. 9], it is possible to prove that a Segal category with finite (co)limits also has (co)limits along any category I whose nerve is has only a finite number of non-degenerate simplices, somewhat justifying the name.

Definition 3.39. [62, pg. 10] A morphism $f : A \to B$ between Segal categories with limits along I is said to *preserve limits along* I if the adjunction-induced morphism

$$f(\lim_I(x_i)) \to \lim_I f(x_i)$$

is an isomorphism in $\mathbf{Ho}(B)$ for any $x_* \in \mathbb{R}\mathrm{Hom}(I, A)$. We say it preserves colimits along I if the dual property for colimits is satisfied.

With this definition in mind, we will produce a somewhat strange definition that we will come to need once we start discussing higher stacks.

Definition 3.40. [62, pg. 10] For Segal categories A and B with finite limits, a morphism $f : A \to B$ in $\mathbf{Ho}(\mathbf{PrSeCat})$ is called *left exact* if it preserves finite limits.

A is called a *left exact localization* of B if there exists a fully faithful morphism $i : A \to B$ in $\mathbf{Ho}(\mathbf{PrSeCat})$ with a left exact left adjoint.

One should see the morphism $i : A \to B$ as an 'inclusion functor' of sorts. We will use this to generalize stackification one we develop a Segal category of suitable higher stacks.

3.5.4 Segal Groupoids

There is a natural notion of a *Segal groupoid*, namely a Segal category A where $\mathbf{Ho}(A)$ is a groupoid [57, pg. 6]. Indeed, at this level, every morphism has an inverse 'up to homotopy'.

The notion of a Segal groupoid gives us an interesting connection between Segal categories and simplicial sets. Consider the *geometric realization* functor $|\cdot| : \mathbf{PrSeCat} \to \mathbf{sSet}$, sending $|A|$ is the simplicial set defined by precomposing A with the diagonal functor on Δ^{op} [57, pg. 6]. This functor actually has a right adjoint $\prod_\infty : \mathbf{sSet} \to \mathbf{PrSeCat}$ defined in [26, pg. 26], where it is called $\prod_{1,se}$ [57, pg. 6]. For a simplicial set X, we will call $\prod_\infty(X)$ its *fundamental Segal groupoid*. This construction actually goes somewhat further: as shown in [45, pg. 301-303], for any Segal groupoid A, the morphism $A \to \prod_\infty(|A|)$ is an equivalence of Segal categories and for any simplicial set X, the morphism $|\prod_\infty(X)| \to X$ is a weak equivalence of simplicial sets.

3.5.5 Model Categories and Segal Categories

There is a rich collaboration between the theories of model categories and Segal categories, which we will exploit here to the best of our abilities. The correspondence between the two is the same as that between model categories and simplicial categories, namely simplicial localization. Any simplicial category is of course a Segal category, so this makes sense.

The conversion from model categories to Segal categories $M \mapsto LM$ can in fact be made functorial with regards to Quillen functors, as suggested by Toën in [57, pg. 7]. Indeed, a left Quillen functor $F : \mathscr{M} \to \mathscr{N}$ between model categories, when restricted to cofibrant objects to give $F^c : \mathscr{M}^c \to \mathscr{N}^c$, preserves weak equivalences, so rather immediately sends hammocks in $\mathscr{M}$ to ones in $\mathscr{N}$. This means a morphism between localizations $LF^c : L^H\mathscr{M}^c \to L^H\mathscr{N}^c$ is induced, which may be translated along the equivalences in 3.30 to get a morphism $LF : L^H\mathscr{M} \to L^H\mathscr{N}$ in $\mathbf{Ho}(\mathbf{PrSeCat})$. This rather self-evidently respects composition.

As an aside, we are now privy to a rather interesting property of model categories that we will make brief use of later, as explained in [57, pg. 7-8]. Any small model category $\mathscr{M}$ admits a Segal category $L\mathscr{M}^{int} \subset L\mathscr{M}$ that is a subset levelwise, defined by the pullback

$$\begin{array}{ccc} L\mathscr{M}^{int} & \longrightarrow & L\mathscr{M} \\ \downarrow & & \downarrow \\ \mathbf{Ho}(\mathscr{M})^{int} & \longrightarrow & \mathbf{Ho}(\mathscr{M}) \end{array}$$

where $\mathbf{Ho}(\mathscr{M})^{int}$ is the maximal subgroupoid of $\mathbf{Ho}(\mathscr{M})$. Toën claims it is possible to show $|L\mathscr{M}^{int}| \simeq |W|$, where W is the subcategory of weak equivalences in $\mathscr{M}$. Hence, we have that $L\mathscr{M}^{int} \simeq \prod_\infty(|W|)$, so $L\mathscr{M}^{int}$ really is in essence the same object as $|W|$. For this reason, we may call $|W|$ the *classifying space* of $\mathscr{M}$, as it models the higher groupoid we get from $\mathscr{M}$ by inverting everything in W.

The connection between model categories and Segal categories runs far deeper than just this, however, which will require us to study localizations more deeply. This starts with a universal property. Consider, for any small category $\mathscr{C}$ with subcategory $S \subset \mathscr{C}$, the morphism $L : \mathscr{C} \to L(\mathscr{C}, S)$ in $\mathbf{Ho}(\mathbf{PrSeCat})$ defined by the diagram

$$\begin{array}{ccc} L(\mathscr{C}, \mathbf{iso}) & \longrightarrow & L(\mathscr{C}, S) \\ \simeq\downarrow & & \\ \mathscr{C} & & \end{array}$$

where $\mathbf{iso}$ is the isomorphisms in $\mathscr{C}$, the horizontal arrow is induced by $1_\mathscr{C}$ and the vertical morphism comes from the adjoint in **S-Cat** to the equivalence of categories $\tau_{\leq 1}L(\mathscr{C}, \mathbf{iso}) \cong \mathscr{C}$.

Proposition 3.35. [62, pg. 13] For every $A \in \mathbf{Ho}(\mathbf{PrSeCat})$, the morphism induced by precomposition

$$L^* : \mathbb{R}\mathrm{Hom}(L(\mathscr{C}, S), A) \to \mathbb{R}\mathrm{Hom}(\mathscr{C}, A)$$

is fully faithful, with an essential image of morphisms $\mathscr{C} \to A$ mapping elements of S to morphisms in A with inverses.

Here, an 'inverse' is simply the Segal category analogue of an inverse in a regular category. This suggests that localization is indeed universal up to equivalence among Segal precategories.

We can quickly deduce an interesting commutativity with products in **Ho**(**PrSeCat**). Consider small categories $\mathscr{C}$ and $\mathscr{D}$, with subcategories $S \subset \mathscr{C}$ and $T \subset \mathscr{D}$. Localization gives us a morphism

$$\begin{aligned} L^*_{C\times D} : &\mathbb{R}\mathrm{Hom}(L(\mathscr{C}\times\mathscr{D}, S\times T), L(\mathscr{C}, S)\times L(\mathscr{D}, T)) \\ &\to \mathbb{R}\mathrm{Hom}(\mathscr{C}\times\mathscr{D}, L(\mathscr{C}, S)\times L(\mathscr{D}, T)) \end{aligned}$$

which is fully faithful and with an essential image sending morphisms in $S\times T$ to invertible morphisms in $L(\mathscr{C}, S)\times L(\mathscr{D}, T)$. This implies that the natural morphism $L_C \times L_D : \mathscr{C}\times\mathscr{D} \to L(\mathscr{C}, S)\times L(\mathscr{D}, T)$ is of the form $L_C \times L_D = M \circ L_{C\times D}$, where M is unique up to equivalence.

Conversely, we may consider

$$\begin{aligned} L^*_{C} : &\mathbb{R}\mathrm{Hom}(L(\mathscr{C}, S), \mathbb{R}\mathrm{Hom}(L(\mathscr{D}, T), L(\mathscr{C}\times\mathscr{D}, S\times T))) \\ &\to \mathbb{R}\mathrm{Hom}(\mathscr{C}, \mathbb{R}\mathrm{Hom}(L(\mathscr{D}, T), L(\mathscr{C}\times\mathscr{D}, S\times T))) \end{aligned}$$

whose codomain can be converted by adjunction into $\mathbb{R}\mathrm{Hom}(\mathscr{C}\times L(\mathscr{D}, T), L(\mathscr{C}\times\mathscr{D}, S\times T))$ and by symmetry and adjunction again into $\mathbb{R}\mathrm{Hom}(L(\mathscr{D}, T), \mathbb{R}\mathrm{Hom}(\mathscr{C}, L(\mathscr{C}\times\mathscr{D}, S\times T)))$. A precomposition again with L_D and one final use of adjunction gives us the codomain $\mathbb{R}\mathrm{Hom}(\mathscr{C}\times\mathscr{D}, L(\mathscr{C}\times\mathscr{D}, S\times T))$. Full faithfulness and the appropriate essential image mean that $L_{C\times D} = N \circ (L_C \times L_D)$, where N is unique up to equivalence. By uniqueness, we must have that M is inverse to N up to equivalence, giving us the following proposition:

Proposition 3.36. [62, pg. 13] For small categories $\mathscr{C}$ and $\mathscr{D}$ as above, the there is an isomorphism in **Ho**(**PrSeCat**) of the form

$$L(\mathscr{C}\times\mathscr{D}, S\times T) \to L(\mathscr{C}, S)\times L(\mathscr{D}, T).$$

With this new weapon in our arsenal, we are prepared to finish our siege on the barrier between model categories and higher category theory. We consider a small category T with subcategory $S \subset T$ and a small cofibrantly generated model category $\mathscr{M}$. Define the category $\mathscr{M}^{(T,S)}$ to be the subcategory of all functors in **Fun**$(T, \mathscr{M})$ sending elements of S to weak equivalences in $\mathscr{M}$. A morphism in this category will be said to be an equivalence if it is an weak equivalence objectwise [58, pg. 20]. With this in mind, we consider the natural *evaluation functor* $\mathscr{M}^{(T,S)}\times T \to \mathscr{M}$ in **PrSeCat**, which sends $W\times S$ to weak equivalences in $\mathscr{M}$. We have an induced morphism in **Ho**(**SeCat**) of the form

$$L(\mathscr{M}^{(T,S)}\times T, W\times S) \simeq L(\mathscr{M}^{(T,S)})\times L(T, S) \to L\mathscr{M}$$

which, by adjunction in **Ho**(**PrSeCat**), becomes a morphism

$$L(\mathscr{M}^T) \to \mathbb{R}\mathrm{Hom}(L(T,S), L\mathscr{M})$$

[62, pg. 13-14]. It turns out that this morphism is actually stronger than we may immediately expect, giving us a statement we will call the *strictification theorem*:

Theorem 3.4. [62, pg. 14] [57, pg. 9] For any small cofibrantly generated model category $\mathscr{M}$ and small category with a subcategory $S \subset T$, the induced morphism

$$L(\mathscr{M}^{(T,S)}) \to \mathbb{R}\mathrm{Hom}(L(T,S), L\mathscr{M})$$

is an isomorphism in **Ho**(**PrSeCat**).

A proof sketch is given in [62, pg. 14] of this crucial theorem. The full proof goes beyond the scope of this dissertation. Also, the assumption of $\mathscr{M}$ being cofibrantly generated will, as with left Bousfield localizations, often be implicit in our usage of this theorem.

The strictification theorem makes it possible to directly translate a number of constructions in model categories to Segal categories. For instance, it is the case that existence of homotopy limits and colimits in a small cofibrantly generated model category $\mathscr{M}$ implies existence of limits and colimits in the Segal category $L\mathscr{M}$ [57, pg. 9]. More benefits of this theorem will become apparent as we apply Segal categories in our hunt for a higher cateogrical analogue of stack theory.

Chapter 4

Higher Stacks

> *"Symmetry, as wide or as narrow as you may define its meaning, is one idea by which man through the ages has tried to comprehend and create order, beauty and perfection."*
>
> – Hermann Weyl

We finally arrive at our overarching goal for this dissertation: the construction of a mathematical object that may describe the geometry inherent in a class of spaces, whose equivalences may be so weak as to not be invertible. In some sense, such cases are the natural extension of symmetry to preservations of structure whose lack of inverse can be attributed to a disregard of unimportant information. For instance, a homotopy may well obliterate the cardinality of a space's underlying set, forbidding any inverse to exist, yet still preserve the homotopical structure that we truly care for. In a similar and in some sense entirely derivative manner, considering equivalence of chain complexes entirely up to (co)homology may well irreparably tarnish some algebraic structure necessary for inversion, but preserving this extra data may well directly conflict with the original moduli problem we posed.

Inspired by our success with higher morphisms to deal with such weak equivalences, the construction we will define in light of this goal is that of a *higher stack*, a kind of stack tracking not only morphisms but higher morphisms between them. A desire for n-stacks had already been voiced by Grothendieck as early as 1983 in his seminal manuscript *Pursuing Stacks* [42], which was later visited by Laumon, Moret-Bailly, Simpson and Walter [51, pg. 1]. All these discussions, however, were done without a concrete formalization for higher stack theory, a step only taken later by Toën and Vezzosi in their papers on homotopical algebraic geometry in 2002 [60] and 2004 [61]. The construction they developed amounts to a special kind of *simplicial presheaf*, an analog to presheaves where the target category is **sSet** rather than **Set**. The growing expectations we may have of this construction thus far are made concrete by a principle from Toën in [57, pg. 4]:

As 1-stacks appear as soon as objects must be classified up to isomorphism, higher stacks appear as soon as objects must be classified up to a notion of equivalence which is weaker than the notion of isomorphism.

Armed with an arsenal of higher category theory, we seek to present a construction of such stacks in full, extending the concept of an Artin stack to this higher context in turn.

4.1 Hypercovers

Our intention for a higher stack is to define a presheaf of some kind, with weak equivalences between elements in its sets, satisfying some suitable notion of descent. As simplicial sets have been so effective thus far in describing weak equivalences, we will choose to focus on the category of *simplicial presheaves* $\mathbf{SPr}(\mathscr{C}) := \mathbf{Fun}(\mathscr{C}^{op}, \mathbf{sSet})$. Each simplicial set will describe families parameterized by the appropriate object in $\mathscr{C}$, with morphisms between them and higher morphisms of all dimensions, respected now by pullback morphisms. This admits a natural model structure, whose equivalences and fibrations are levelwise [60, pg. 13].

What kind of descent condition should we really expect from a simplicial presheaf? Recall in the case of stacks that the amalgamation functor had to only be an equivalence of categories, meaning descent data needed to be effective *up to isomorphism*. Indeed, we should expect descent data to be effective *up to weak equivalence* on a simplicial presheaf by this logic, as this is the notion of equality we really care for. We define a special simplicial object to help is in this regard:

Definition 4.1. [10, pg. 2] Let $\mathscr{C}$ be a site with all fibre products and a covering $U_i \to X$. The *Čech nerve* $\check{C}U_*$ is the simplicial object in $\mathscr{C}$ such that

$$\check{C}U_n = \coprod_{a_0 \ldots a_n} U_{a_0} \times_X \cdots \times_X U_{a_n}$$

with simplicial maps defined by projections and duplications of one term.

This is a somewhat natural generalization of the descent data we have seen for sheaves and stacks and should immediately remind the astute algebraic geometer of Čech cohomology. With this, we may declare the descent condition on simplicial presheaves to be that for any simplicial presheaf $F \in \mathbf{SPr}(\mathscr{C})$ and any covering $U_i \to X$, the natural morphism in $\mathbf{Ho}(\mathbf{sSet})$

$$F(X) \to \mathbb{R}\lim F(\check{C}U_*)$$

is an isomorphism. One should note that if $F(A)$ is discrete for every A, then this is just the sheaf condition, becoming the stack condition if $F(A)$ is instead always the nerve of a small category.

This seems to be a suitable extension of the descent condition on sheaves and stacks to weak equivalence and would essentially give us a concrete implementation of the notion of an ∞-*stack*, as defined by Lurie in terms of his idealized notion of an '∞-category' that Segal categories, simplicial categories and other models aim to implement [30, pg. 34]. However, Toën and Vezzosi instead go one step further and impose a stricter descent condition called *hyperdescent*, giving us what Lurie calls the ∞-*hyperstacks* [30, pg. 40]. Why would we need this? Unfortunately, if we try to apply the descent condition above in terms of Segal categories and then in terms of a model category of simplicial presheaves, the former turns out not to be the localization of the latter due to fundamental homotopical discrepancies [30, pg. 37], meaning we lack the duality of an intrinsic Segal category and useful model category presentation. We will need a more general kind of covering to descend upon, known as a *hypercover*, if this issue is to be rectified.

Our motivation for hypercovers so far feels somewhat weak and evasive of what are potentially fundamental issues with trying to produce a higher stack, so I will try to provide some intuition here. Consider an object $X \in \mathscr{C}$ and a simplicial presheaf $F \in \mathbf{SPr}(\mathscr{C})$. The object $F(X)$ is of course a simplicial set, so a descent condition should consist of some construction $U_* \to X$ derived from a covering of X in $\mathscr{C}$ such that, informally, the induced map $F(X) \to \mathbb{R}\mathrm{lim} F(U_*)$ is a weak equivalence. We select a covering $U_i \to X$ in $\mathscr{C}$. We would like each 0-simplex in $F(X)$ to correspond uniquely with a suitable 0-simplex in $\prod_i F(U_i)$. What should 'suitable' mean here? Clearly we are only interested in elements of $\prod_i F(U_i)$ that agree on fibre products $\prod_{i,j} F(U_i \times_X U_j)$. In fact, we could weaken this even further by noting they need only agree *locally* on fibre products, ie. on covers $U_{ij} \to U_i \times_X U_j$. One could imagine repeating this for higher and higher dimensions of fibre product, essentially giving covers for each n-fold intersection in $U_i \to X$. This is the idea of a hypercover.

More formally, consider some object in a site $X \in \mathscr{C}$, viewed as a presheaf. We may see this as a simplicial presheaf where each simplicial set in the image is discrete. Our starting point is then a simplicial presheaf U_* where after converting by adjunction to a functor $\Delta^{op} \to \mathbf{Ps}(\mathscr{C})$ each level U_n is a coproduct of representable presheaves, together with a morphism $U_* \to X$ of simplicial presheaves. If for $F \in \mathbf{SPr}(\mathscr{C})$ and $K \in \mathbf{sSet}$ we define F^K to be the simplicial presheaf such that $F^K(A) = \mathbf{Map}(K, F(A))$ for all A and furthermore set $M_n F = (F^{\partial\Delta^n})_0$, we see that a Čech nerve may be identified by demanding that $U_0 \to X$ is the coproduct of a cover and the maps $U_n \to M_n U$ induced by the natural maps $U^{\Delta^n} \to U^{\partial\Delta^n}$ for all $n \geq 1$ are isomorphisms [10, pg. 8]. The definition we have given of a Čech nerve follows by induction.

We want to weaken these isomorphisms to some analogue to a covering. The following definition lets us do this:

Definition 4.2. [10, pg. 7] A morphism between simplicial presheaves $f : F \to G$ is a *generalized covering*, or *local epimorphism*, if given any map $h : X \to G$, there is a covering sieve $U_i \to X$ such that for each constituent morphism $g_i : U_i \to X$ the composite $h \circ g_i : U_i \to X \to G$ lifts through f.

It is not hard to see that, if F is a coproduct of representables and G is representable itself, then $F \to G$ is a generalized covering exactly when it is the coproduct of all maps in a covering sieve of G [10, pg. 7].

With this in mind, a weakening of the alternative definition we have given of a Čech nerve such that each map $U_n \to M_n U$ is a generalized covering for $n \geq 1$ will give us a hypercover. We state this in complete detail:

Definition 4.3. [10, pg. 8] Let $X \in \mathscr{C}$ and $U_* \in \mathbf{SPr}(\mathscr{C})$ with a map $U_* \to X$, where X is seen as a trivial simplicial presheaf. Then U_* is called a *hypercover* if each U_n is a coproduct of representables and the maps $U_0 \to X$, $U_1 \to U_0 \times_X U_0$ and $U_n \to M_n U_*$ are all coverings.

With hypercovers in place, we can now state formally what it means to descend upon them. The definition is as would be expected:

Definition 4.4. [10, pg. 8] Let $F \in \mathbf{SPr}(\mathscr{C})$. We say that F *satisfies hyperdescent* if for every hypercover $U_* \to X$ where $U_n = \coprod_i U_i^n$ for every n, the induced map in $\mathbf{Ho}(\mathbf{sSet})$

$$F(X) \longrightarrow \mathbb{R}\mathrm{lim}\Big(\prod_i F(U_i^0) \rightrightarrows \prod_i F(U_i^1) \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \cdots \Big)$$

is an isomorphism.

If we are willing to relax the requirement that each U_n is a coproduct of representables for a moment, we find a rather intriguing alternative interpretation of hypercovers that we will make use of later. For a general morphism $U \to F$ in $\mathbf{SPr}(\mathscr{C})$, define

$K \otimes F \in \mathbf{SPr}(\mathscr{C})$ for $K \in \mathbf{sSet}$ as $(K \otimes F)(X) = K \times F(X)$. Now for some $X \in \mathscr{C}$, consider a lifting problem of the form

$$\begin{array}{ccc} \partial\Delta^n \otimes X & \longrightarrow & U \\ \downarrow & & \downarrow \\ \Delta^n \otimes X & \longrightarrow & F \end{array}$$

as seen in [10, pg. 7]. The bottom horizontal map may be seen as a map of presheaves $X \to F_n$, or an element of $F_n(X)$. If these lifting problems admitted a solution for all $n \geq 0$, we could by induction lift any element of $F_n(X)$ to $U_n(X)$, meaning anything in $F(X)$ could be lifted to $U(X)$. However, we are not particularly interested in such a direct lift; rather, we would like a *local lifting*, by which we mean there is a covering $U_i \to X$ of X such that the diagram obtained by restricting from X to each U_i has a lifting $\Delta^n \otimes U_i \to F$ [10, pg. 7]. Informally, U surjects 'locally' onto X.

Yet another way of saying this is that the induced map

$$U_n \to M_n U \times_{M_n F} F_n$$

is a generalized covering for every $n \geq 0$ [10, pg. 8], which gives a clean generalization of hypercovers. After all, the hypercovers we have defined are exactly those where F is representable and every U_n is a coproduct of representables.

4.2 Higher Stacks

There are several ways to go about defining the higher stacks over a site $\mathscr{C}$. If we consider the simpler case of the category of sheaves $\mathbf{Sh}(\mathscr{C})$, Toën reveals a rather elegant approach through universal properties [57, pg. 10]. To this end, we will write $\mathbf{Fun}_c(\mathscr{C}, \mathscr{D})$ for the category of functors commuting with colimits. He notes that, up to equivalence, this category is one with a functor $h : \mathscr{C} \to \mathbf{Sh}(\mathscr{C})$ such that:

1. $\mathbf{Sh}(\mathscr{C})$ has all colimits.

2. For any category with colimits $\mathscr{D}$, the induced functor

$$h^* : \mathbf{Fun}_c(\mathbf{Sh}(\mathscr{C}), \mathscr{D}) \to \mathbf{Fun}(\mathscr{C}, \mathscr{D})$$

   is fully faithful and has an image consisting of all functors $F : \mathscr{C} \to \mathscr{D}$ such that for any object $X \in \mathscr{D}$ and any hypercover $U_* \to X$, the induced morphism in $\mathscr{D}$

$$\mathrm{colim}_{\overline{n} \in \Delta^{op}} F(U_n) \to F(X)$$

   is an isomorphism.

There is an implicit abuse of notation here, since each U_n is a coproduct of representable presheaves over $\mathscr{C}$ and should be seen as a formal disjoint union of objects in $\mathscr{C}$ [57, pg. 10]. For a formal disjoint union $U = \coprod U_i$, the notation $F(U)$ stands for $\coprod F(U_i)$[1].

We are now ready to define our concept of higher stacks:

Definition 4.5. [57, pg. 10-11] Given a site $\mathscr{C}$, a *Segal category of stacks* is a Segal category $\mathbf{St}(\mathscr{C})$ together with a functor $h : \mathscr{C} \to \mathbf{St}(\mathscr{C})$ such that:

1. The category $\mathbf{St}(\mathscr{C})$ has all colimits.

2. For any Segal category with colimits $\mathscr{D}$, the induced functor

$$h^* : \mathbb{R}\mathrm{Hom}_c(\mathbf{St}(\mathscr{C}), \mathscr{D}) \to \mathbb{R}\mathrm{Hom}(\mathscr{C}, \mathscr{D})$$

is fully faithful and has an image consisting of all functors $F : \mathscr{C} \to \mathscr{D}$ such that for any object $X \in \mathscr{C}$ and any hypercover $U_* \to X$ in $\mathscr{C}$, the induced morphism in $\mathscr{D}$

$$\mathrm{colim}_{\overline{n} \in \Delta^{op}} F(U_n) \to F(X)$$

is an isomorphism in $\mathbf{Ho}(\mathscr{D})$.

We have of course taken $\mathbb{R}\mathrm{Hom}_c(A, B)$ to mean those morphisms that commute with colimits.

The uniqueness up to equivalence of such a Segal category is not hard to see. To prove existence, we will have to work somewhat harder. This can be done either by directly working with candidate Segal categories or a model presentation.

4.2.1 The Model Category Approach

Our first direction of attack will be to define a suitable model category and localize it. We recall the category of simplicial presheaves over $\mathscr{C}$, denoted $\mathbf{SPr}(\mathscr{C})$, with its model structure where weak equivalences and fibrations are defined levelwise [60, pg. 13]. However, this category enjoys yet another important model structure, defined to be a left Bousfield localization by a class of morphisms representative of hypercovers. This will require some technical machinery to define.

Before we venture further down this path, a point must be made about the potential simplicial structure on $\mathbf{SPr}(\mathscr{C})$. The model category $\mathbf{sSet}$ may be made into

[1] Toën in [57, pg. 10] writes $\prod$ instead of $\coprod$ here, which I believe to be an error arising from the case of hyperdescent.

a *simplicial model category*, namely a simplicial category whose underlying category has a suitably compatible model structure, as defined in [33, ch. 9]. This induces a similar structure on $\mathbf{SPr}(\mathscr{C})$ [60, pg. 13]. However, as any simplicial model category's full simplicial subcategory of fibrant and cofibrant objects is in fact equivalent to the localization of its underlying model category [60, pg. 12] and Toën and Vezzosi seem to pay this simplicial structure little attention, this will not bother us so much, so we will ignore it for simplicity's sake.

We first define a covering in $\mathbf{SPr}(\mathscr{C})$ as per [60]. The functor π_0 naturally defines, by postcomposition, a functor $\pi_0^{ps} : \mathbf{SPr}(\mathscr{C}) \to \mathbf{Ps}(\mathscr{C})$ sending any simplicial presheaf to its presheaf of connected components. Since $\mathscr{C}$ is in fact a site with topology J, we can sheafify any $\pi_0^{ps}(F)$ to get $\pi_0^J(F)$, the *sheaf of connected components* of $F \in \mathbf{SPr}(\mathscr{C})$ [60, pg. 19]. A morphism $F \to G$ in $\mathbf{Ho}(\mathbf{SPr}(\mathscr{C}))$ is then called a *J-covering*, or just a covering if J is obvious, if the induced morphism $\pi_0^J(F) \to \pi_0^J(G)$ is an epimorphism of sheaves [60, pg. 19]. This, in essence, defines a covering 'up to homotopy'. We could then say a morphism in $\mathbf{SPr}(\mathscr{C})$ is a covering if its representative in the homotopy category is as such.

Define $\mathbf{sSPr}(\mathscr{C}) := \mathbf{SPr}(\mathscr{C})^{\Delta^{op}}$ to be the category of simplicial objects in $\mathbf{SPr}(\mathscr{C})$. We denote its objects by F_*, where F_m is the simplicial presheaf $F_*(\overline{m})$. The simplicial presheaves have all limits and colimits, so there is a tensor and co-tensor structure on $\mathbf{sSPr}(\mathscr{C})$ over $\mathbf{sSet}$ [60, pg. 20], by Theorem 2.5 in [20, pg. 87]. The tensor product structure, for $F_* \in \mathbf{sSPr}(\mathscr{C})$ and $A \in \mathbf{sSet}$, takes the form

$$\begin{aligned} \underline{A} \otimes F_* : \Delta^{op} &\to \mathbf{Spr}(\mathscr{C}) \\ \overline{n} &\mapsto \coprod_{A_n} F_n \end{aligned}$$

The co-tensor, or exponential, of F_* by A is denoted by $F_*^{\underline{A}}$ and is given by the adjunction isomorphism

$$\mathrm{Hom}(\underline{A} \otimes F_*, G_*) \simeq \mathrm{Hom}(F_*, G_*^{\underline{A}})$$

[60, pg. 20-21]. With this in mind, we will write $F_*^A \in \mathbf{SPr}(\mathscr{C})$ to mean $(F^{\underline{A}})_0$. It is in fact the case that there is a natural isomorphism $F_*^{\Delta^n} \simeq F_n$ for all $n \geq 0$ and $F_* \in \mathbf{sSPr}(\mathscr{C})$ [60, pg. 21].

An interesting fact is that this can actually be directly done in $\mathbf{SPr}(\mathscr{C})$ as well; since $\mathbf{SPr}(\mathscr{C}) \cong \mathbf{Fun}(\Delta^{op}, \mathbf{Ps}(\mathscr{C}))$ naturally by the Cartesian closed structure on $\mathbf{Cat}$ [31, pg. 98] and presheaves have all limits and colimits pointwise, we can apply the above reasoning again to get an exponential object F^A for any $F \in \mathbf{SPr}(\mathscr{C})$ and $A \in \mathbf{sSet}$.

There is a model structure called the *Reedy model structure* on $\mathbf{sSPr}(\mathscr{C})$, where the weak equivalences are levelwise [60, pg. 21]. It is possible to show that the functor

$$(-)^A : \mathbf{sSPr}(\mathscr{C}) \to \mathbf{SPr}(\mathscr{C})$$

is right Quillen with regards to this model structure, which implies it has a right derived functor

$$(-)^{\mathbb{R}A} : \mathbf{Ho}(\mathbf{sSPr}(\mathscr{C})) \to \mathbf{Ho}(\mathbf{SPr}(\mathscr{C}))$$

[60, pg. 21].

This can be applied to any $F \in \mathbf{SPr}(\mathscr{C})$ by constructing the constant simplicial object $c(F)_* \in \mathbf{sSPr}(\mathscr{C})$ defined by $c(F)_n = F$ for all n to get $(c(F)_*)^{\mathbb{R}A}$. To understand this object, let us define $(RF)^{\Delta^*} \in \mathbf{sSPr}(\mathscr{C})$ such that $(RF)^{\Delta^*}_n = (RF)^{\Delta^n}$, where RF is a fibrant replacement of F. It is the case that, for any $A \in \mathbf{sSet}$, we have a natural isomorphism $((RF)^{\Delta^*})^A \simeq (RF)^A$ [60, pg. 21]. From this, one may derive a series of natural isomorphisms $(c(F)_*)^{\mathbb{R}A} \simeq ((RF)^{\Delta^*})^A \simeq (RF)^A$ in $\mathbf{Ho}(\mathbf{SPr}(\mathscr{C}))$ [60, pg. 21]. This is not, however, an isomorphism between $c(F)^A_*$ and F^A in $\mathbf{SPr}(\mathscr{C})$ - it is only up to homotopy. We will write $F^{\mathbb{R}A}$ for $F \in \mathbf{SPr}(\mathscr{C})$ to mean $(c(F)_*)^{\mathbb{R}A}$ from now on.

Definition 4.6. [60, pg. 23] A morphism $F_* \to G_*$ in $\mathbf{sSPr}(\mathscr{C})$ is called a *J-hypercover*, or hypercover if J is clear, if for any $n \geq 0$ the induced morphism

$$F_*^{\mathbb{R}\Delta^n} \simeq F_n \to F_*^{\mathbb{R}\partial\Delta^n} \times^h_{G_*^{\mathbb{R}\partial\Delta^n}} G_*^{\mathbb{R}\Delta^n}$$

is a covering in $\mathbf{Ho}(\mathbf{SPr}(\mathscr{C}))$, where $- \times^h_- -$ is the homotopy limit of a cospan, ie. a 'homotopy pullback'. A morphism in $\mathbf{Ho}(\mathbf{sSPr}(\mathscr{C}))$ is called a hypercover if one of its representatives is as such.

With this in mind, we will finally define $\mathbf{SPr}_J(\mathscr{C})$, the *model category of stacks.* to be the left Bousfield localization of $\mathbf{SPr}(\mathscr{C})$ along morphisms $|F_*| \to RX$ induced by hypercovers $F_* \to (RX)^{\Delta^*}$ in $\mathbf{Ho}(\mathbf{sSPr}(\mathscr{C}))$, for all $X \in \mathscr{C}$ [60, pg. 32]. The fibrant replacement RX is of course in $\mathbf{SPr}(\mathscr{C})$.

We can actually give a concrete definition for a 'stack' in $\mathbf{SPr}_J(\mathscr{C})$:

Definition 4.7. [60, pg. 33] A hypercover $H_* \to X$ with $X \in \mathscr{C}$ is said to be *semi-representable* if for any $n \geq 0$, H_n is isomorphic in $\mathbf{Ho}(\mathbf{SPr}(\mathscr{C}))$ to a coproduct of representable objects

$$H_n \simeq \coprod_{u \in I_n} u$$

An object $F \in \mathbf{SPr}(\mathscr{C})$ is then said to satisfy *hyperdescent* if, for any object $X \in \mathscr{C}$ and semi-representable hypercover $H_* \to X$, the natural morphism

$$F(X) \to \mathbb{R}\mathrm{lim}_{\overline{n} \in \Delta} \left(\prod_{u \in I_n} F(u) \right)$$

is an equivalence of simplicial sets.

A *stack* is any $F \in \mathbf{SPr}(\mathscr{C})$ satisfying hyperdescent.

This seems to coincide with our original discussions of hypercovers somewhat better; for starters, we've finally employed hypercovers comprised of coproducts of Yoneda embeddings. Furthermore, the specific descent condition stacks satisfy is now much clearer. At this point, if we consider the left Quillen identity functor $\mathbf{SPr}(\mathscr{C}) \to \mathbf{SPr}_J(\mathscr{C})$ induced by Bousfield localization, it can be proven that its right adjoint induces a fully faithful functor on homotopy categories whose image is exactly the stacks [60, pg. 34].

4.2.2 The Segal Category Approach

Another path to higher stacks will be one that directly produces an appropriate Segal category, largely following Toën in [62, pg. 8]. This begins by constructing a higher category theoretic version of a presheaf, in essence identical to simplicial presheaves. However, we choose to employ a different target category than **sSet**, instead opting for the Segal category

$$\mathrm{Top} := L\mathbf{sSet}$$

consisting of simplicial sets with higher morphisms encoding the weak homotopy equivalences [57, pg. 7]. This structure is somewhat evidently a combinatorial version of topological spaces, encoding the homotopy of CW complexes, so the name is well-deserved.

Considering $\mathscr{C}$ as a Segal category itself, we may define the *category of prestacks*

$$\hat{\mathscr{C}} := \mathbb{R}\mathrm{Hom}(\mathscr{C}^{op}, \mathrm{Top})$$

assigning simplicial sets to various elements of the category $\mathscr{C}$. To fully cement this as a higher version of a presheaf, we define a Yoneda embedding, using a simplification of the definition in [62, pg. 14]. For a category $\mathscr{C}$ considered as an simplicial category, we have the natural functor $\mathscr{C}^{op} \times \mathscr{C} \to \mathbf{sSet}$ sending a pair of objects to their hom-set, with morphisms becoming composition. Adjunction in categories makes this a functor $\mathscr{C} \to \mathbf{Fun}(\mathscr{C}^{op}, \mathbf{sSet})$, which by strictification gives us a morphism of Segal categories

$$h : \mathscr{C} \to \hat{\mathscr{C}}$$

well-defined in $\mathbf{Ho}(\mathbf{PrSeCat})$. By the classical Yoneda lemma and strictification, this will be fully faithful. This is indeed possible to perform where $\mathscr{C}$ is in fact a Segal category by first taking an equivalent replacement with a simplicial category and invoking a simplicial category Yoneda lemma [62, pg. 14], though we will not need this here. It is predictable that we then get the following result:

Lemma 4.1. (Yoneda Lemma) [62, pg. 14] Let $\mathscr{C}$ be a category with $X \in \mathrm{Ob}(\mathscr{C})$ and $F \in \hat{\mathscr{C}}$. There is a natural equivalence of simplicial sets

$$F(X) \simeq \hat{\mathscr{C}}_{(h_X, F)}.$$

We will call an element of $\hat{\mathscr{C}}$ *representable* if it is equivalent to h_X for some $X \in \mathrm{Ob}(\mathscr{C})$.

As it stands, while a step in the right direction, a higher prestack is insufficient for our purposes; it needs some descent condition to truly be considered reminiscent of a stack. We presume $\mathscr{C}$ to be a site with a topology J. The functor $\pi_0 : \mathbf{sSet} \to \mathbf{Set}$, sending a simplicial set to its path components, implies a functor $\pi_0 : \mathrm{Top} \to \mathbf{Set}$ via localization. Postcomposition yields a new functor

$$\pi_0^{pr} : \hat{\mathscr{C}} \to \mathbb{R}\mathrm{Hom}(\mathscr{C}, \mathbf{Set}) \simeq \mathbf{Ps}(\mathscr{C})$$

[62, pg. 10]. Similarly, given a simplicial set K, the functor $\mathbf{Map}(K, -) : \mathbf{sSet} \to \mathbf{sSet}$ induces a functor

$$(-)^{\mathbb{R}K} : \mathrm{Top} \to \mathrm{Top}$$

when restricted to fibrant objects, which then induces a similar functor by postcomposition

$$(-)^{\mathbb{R}K} : \hat{\mathscr{C}} \to \hat{\mathscr{C}}$$

[62, pg. 10]. We are now ready to propose a more concrete definition of a higher stack.

Definition 4.8. [62, pg. 10] Let $\mathscr{C}$ be a site with topology J.

1. A morphism $f : F \to G$ in $\hat{\mathscr{C}}$ is a *J-local equivalence* if and only if, for any integer $n \geq 0$, the induced morphism in $\mathbf{Ps}(\mathscr{C})$

$$\pi_0^{pr}(F^{\mathbb{R}\Delta^n}) \to \pi_0^{ps}(F^{\mathbb{R}\partial\Delta^n} \times_{G^{\mathbb{R}\partial\Delta^n}} G^{\mathbb{R}\Delta^n})$$

induces an epimorphism of sheaves on $\mathbf{Sh}(\mathscr{C})$.

2. An object $F \in (\hat{\mathscr{C}})_0$ is a *stack* on the topology J if and only if, for any J-local equivalence $f : G \to H$ in $\hat{\mathscr{C}}$, the induced morphism

$$f^* : \mathrm{Hom}_{\mathbf{Ho}(\hat{\mathscr{C}})}(H, F) \to \mathrm{Hom}_{\mathbf{Ho}(\hat{\mathscr{C}})}(G, F)$$

is a bijection.

The *Segal category of stacks* $\mathscr{C}^{\sim,J}$ on the site $\mathscr{C}$ with topology J is the full subcategory of $\hat{\mathscr{C}}$ whose objects are stacks on the topology J.

It is not hard to rephrase this definition so it better aligns with the notion of hypercover we have seen. In some sense, the J-local equivalences are hypercovers 'up to path components'. At the moment, this is just a vague interpretation, one which we will aim to formalize. This begins with the notion of a *Segal topos*:

Definition 4.9. [62, pg. 11] A *Segal topos* is a Segal category which is a left exact localization of $\hat{\mathscr{C}}$ for some Segal category $\mathscr{C}$.

As we have already noted, $\hat{\mathscr{C}}$ can be defined for a general Segal category $\mathscr{C}$ just as it can for categories. However, we will not be interested in this generalized case so much. Because a Segal topos A always comes with a Segal category $\mathscr{C}$ and a morphism $i : A \to \hat{\mathscr{C}}$ which is fully faithful and in possession of a left exact left adjoint by its definition [62, pg. 11], it is possible to show that any Segal topos has all limits and colimits [62, pg. 11].

For a Segal category A, an object $y \in A$ will be called *n-truncated* if the simplicial set $A_{(x,y)}$ is n-truncated, meaning it is equivalent to its n-skeleton, for any $x \in A$. We may then declare y to be *truncated* if it is n-truncated for some n [62, pg. 11].

Definition 4.10. [62, pg. 11] A Segal category A is *t-complete* if a morphism $f : X \to Y$ in A is an isomorphism in $\mathbf{Ho}(A)$ exactly when $f^* : \mathrm{Hom}_{\mathbf{Ho}(A)}(Y, B) \to \mathrm{Hom}_{\mathbf{Ho}(A)}(X, B)$ is bijective for any truncated $B \in \mathbf{Ho}(A)$.

Proposition 4.1. [62, pg. 11] Let $\mathscr{C}$ be a site with topology J. The natural inclusion functor $\mathscr{C}^{\sim,J} \to \hat{\mathscr{C}}$ has a left exact left adjoint, making $\mathscr{C}^{\sim,J}$ a Segal topos. Furthermore, it is t-complete.

This proof of this derives from work in [60] proving a version of this proposition with respect to model categories and applying a correspondence we will see later.

The fact that our notion of a stack forms a t-complete Segal topos suggests a few important points. The first is that we have a natural inclusion of stacks into prestacks, with a left adjoint that will represent *stackification*. Furthermore, all colimits exist, satisfying the first part of our universal definition for higher stacks.

Why should we see t-completeness as a predictable property? It seems to be somewhat artificial, so we will immediately replace it with an equivalent property based on hypercovers that better represents our ambitions for $\mathscr{C}^{\sim,J}$. Define the nerve of a morphism $f : F \to G$ between Segal categories, written $N(f)$, to be the simplicial object of the form [30, pg. 21]

$$\cdots \; F \times_G F \times_G F \; F \times_G F \; F$$

We will write $|X_*|$ to be the colimit of such a simplicial object X_*, a nod to the geometric realization functor we defined in the simplicial set case. We will choose to call a morphism $f : F \to G$ in a Segal topos A an *epimorphism* exactly when the induced morphism $|N(f)| \to G$ is an equivalence [62, pg. 16]. One should think of $N(f)$ defining an abstract simplicial complex, where each fibre of f has any n $(n-1)$-simplices connected by an n-simplex and any two fibres completely disjoint. The induced morphism's fibres are now each a complex with trivial homotopy type, which is of course surjective exactly when the map is an equivalence. The reader should note that this is essentially identical to a covering as defined in the model category case.

Consider the subcategory $(\mathbf{SPr}(\mathscr{C})^f)^{(\mathbf{sSPr}(\mathscr{C})^f, W^f)}$ of functors with domain and codomain restricted to fibrant objects, where W^f is the levelwise weak equivalences between fibrant objects in $\mathbf{sSPr}(\mathscr{C})$. Strictification gives us

$$\begin{aligned} L(\mathbf{SPr}(\mathscr{C})^f)^{(\mathbf{sSPr}(\mathscr{C})^f, W^f)} &\simeq \mathbb{R}\mathrm{Hom}(L(\mathbf{sSPr}(\mathscr{C})^f, W^f), \hat{\mathscr{C}}) \\ &\simeq \mathbb{R}\mathrm{Hom}(\mathbb{R}\mathrm{Hom}(\Delta^{op}, \hat{\mathscr{C}}), \hat{\mathscr{C}}) \end{aligned}$$

Along these equivalences, we may transmit $(-)^K : \mathbf{sSPr}(\mathscr{C}) \to \mathbf{SPr}(\mathscr{C})$, since it is right Quillen and thus preserves fibrations and acyclic fibrations. This gives us a functor $(-)^{\mathbb{R}K} : \mathbb{R}\mathrm{Hom}(\Delta^{op}, \hat{\mathscr{C}}) \to \hat{\mathscr{C}}$. With this in mind, we may declare a hypercover in A to be a morphism $X_* \to Y$ between simplicial objects in $\hat{\mathscr{C}}$, where Y is a constant simplicial object, such that for any $n \geq 0$ the induced morphism

$$X_*^{\mathbb{R}\Delta^n} \to X_*^{\mathbb{R}\partial\Delta^n} \times_{Y^{\mathbb{R}\partial\Delta^n}} Y^{\mathbb{R}\Delta^n}$$

is an epimorphism [62, pg. 17][2].

[2]Part of this is my own interpretation; Toën and Vezzosi have not clarified in [62] what $X_*^{\mathbb{R}K}$ should mean for a simplicial object X_* in a Segal topos. I believe, due to the correspondence with the model category case we will soon see, that what I have written is correct as it correlates well with the model category definition of a hypercover.

Proposition 4.2. [62, pg. 17] A Segal topos A is t-complete exactly when for any hypercover $X_* \to Y$, the induced morphism $|X_*| \to Y$ is an equivalence.

While we choose not to give a proof of this proposition, it is somewhat self-evident now how the categories $\mathscr{C}^{\sim,J}$ satisfy the requirements of a stack. The correspondence is actually even better:

Proposition 4.3. [62, pg. 11] The mapping $J \to \mathscr{C}^{\sim,J}$ produces a bijection between topologies on $\mathscr{C}$ and equivalence classes of t-complete left exact localizations of $\hat{\mathscr{C}}$.

We will not prove this result here - it derives from the comparison with the model category case we will explore soon, along with the corresponding case in [60, pg. 39]. With this, it becomes clear that our current notion of stacks is exactly characterized by the left exact localizations of prestacks where hypercovers induce equivalences.

It is at long last that we may forge the expected connection between the Segal category and model category approaches to higher stack theory:

Theorem 4.1. [62, pg. 15] For any site $\mathscr{C}$ with topology J, there exists natural isomorphisms in $\mathbf{Ho}(\mathbf{PrSeCat})$

$$L(\mathbf{SPr}(\mathscr{C})) \simeq \hat{\mathscr{C}} \qquad L(\mathbf{SPr}_J(\mathscr{C})) \simeq \mathscr{C}^{\sim,J}$$

The proof of this relies on the fact that $\mathbf{SPr}_J(\mathscr{C})$ is a so-called *model topos*, a model category Quillen equivalent to a left Bousfield localization of $\mathbf{SPr}(\mathscr{C})$ such that the localization functor preserves homotopy pullbacks [60, pg. 39]. One then establishes that any model topos has a localization which is a Segal topos [62, pg. 15] and that $\mathbf{SPr}_J(\mathscr{C})$ satisfies a model category version of t-completeness exactly analogous to the Segal category case [60, pg. 41]. The result then follows.

While we will not complete it, we can begin to explore proving the universal definition of higher stacks we proposed earlier. By Proposition 3.35, the localization morphism $L : \mathbf{SPr}(\mathscr{C}) \to \mathscr{C}^{\sim,J}$ is such that for any Segal category with colimits B, the induced morphism

$$L^* : \mathbb{R}\mathrm{Hom}(\mathscr{C}^{\sim,J}, B) \to \mathbb{R}\mathrm{Hom}(\mathbf{SPr}(\mathscr{C}), B)$$

is fully faithful with an essential image of functors $F : \mathbf{SPr}(\mathscr{C}) \to B$ such that if $U \to X$ is a weak equivalence in $\mathbf{SPr}_J(\mathscr{C})$ then $F(U) \to F(X)$ is an isomorphism in $\mathbf{Ho}(B)$. If we restrict this to functors preserving colimits and compose with the Yoneda embedding, then we get a morphism

$$h^* \circ L^* : \mathbb{R}\mathrm{Hom}_c(\mathscr{C}^{\sim,J}, B) \to \mathbb{R}\mathrm{Hom}(\mathscr{C}, B)$$

which is fully faithful and whose essential image is a subclass of the functors $F : \mathscr{C} \to B$ such that for any hypercover $U_* \to X$ in $\mathscr{C}$, the induced morphism of the form $\mathrm{colim}_{\overline{n}\in\Delta^{op}} F(U_n) \to F(X)$ is an isomorphism in $\mathbf{Ho}(B)$, where $F(U_n)$ represents the same abuse of notation we have applied before. This brings us closer to the universal characterization of a Segal category of stacks, though our image is not every such functor; solving that will be left as future work, as there seems not to be a complete proof directly given in the literature.

As a final point of interest, recall the fully faithful functor we discussed in the model category case whose image in homotopy categories was exactly the stacks. When we elevate these functors to localizations, we see that this must in fact exactly coincide with the notion of stack in Segal categories.

4.2.3 Stacks of Morphisms

With our formalized concept of a higher stack in tow, our new aim is to try and produce some useful stack constructions. The first one we will consider is that of a *mapping stack*, a stack of morphisms between two stacks F and G. This will require a bit of toying with the model category structure $\mathbf{SPr}_J(\mathscr{C})$ we've defined.

The first thing to note is that a morphism $F \to G$ in $\mathbf{SPr}_J(\mathscr{C})$ is a weak equivalence exactly when the induced morphism $c(F)_* \to c(G)_*$ is a J-hypercover [60, pg. 24]. It is possible to replace this model structure with the *injective model structure* $\mathbf{SPr}_{inj,J}(\mathscr{C})$, with the same weak equivalences but whose cofibrations are now the monomorphisms [60, pg. 35]. Furthermore, since weak equivalences and cofibrations in $\mathbf{SPr}_{inj,J}(\mathscr{C})$ are closed under finite products [60, pg. 35] and every object is cofibrant, we may declare $\mathbf{SPr}_{inj,J}(\mathscr{C})$ to be something called a *monoidal model category* [28, pg. 109], which implies, as we have seen before with Segal precategories, that the homotopy category $\mathbf{Ho}(\mathbf{SPr}_{inj,J}(\mathscr{C}))$ is Cartesian closed [60, pg. 35]. However, since this model structure has the same weak equivalences as before, we have

$$\mathbf{Ho}(\mathbf{SPr}_{inj,J}(\mathscr{C})) \cong \mathbf{Ho}(\mathbf{SPr}_J(\mathscr{C}))$$

meaning we can define an internal structure on the homotopy category of stacks:

Definition 4.11. [60, pg. 36] The *internal hom's* on the category $\mathbf{Ho}(\mathbf{SPr}_J(\mathscr{C}))$ shall be written as

$$\mathbb{R}_J\mathbf{Hom}(-,-) : \mathbf{Ho}(\mathbf{SPr}_J(\mathscr{C}))^{op} \times \mathbf{Ho}(\mathbf{SPr}_J(\mathscr{C})) \to \mathbf{Ho}(\mathbf{SPr}_J(\mathscr{C}))$$

sending any $F, G \in \mathbf{Ho}(\mathbf{SPr}_J(\mathscr{C}))$ to their *stack of morphisms* $\mathbb{R}_J\mathbf{Hom}(F, G)$. More explicitly, by [28, pg. 116] we have

$$\mathbb{R}_J\mathbf{Hom}(F, G) \simeq \mathbf{Hom}(F, R_{inj}G)$$

where the latter object is the internal hom in $\mathbf{SPr}(\mathscr{C})$ [60, pg. 36] and $R_{inj}G$ is a fibrant replacement for G in $\mathbf{SPr}_{inj,J}(\mathscr{C})$. Both $R_{inj}G$ and $\mathbf{Hom}(F, R_{inj}G)$ are stacks exactly when G is a stack.

Of course, we rather immediately have that $\mathbf{Ho}(\mathbf{SPr}_J(\mathscr{C})) \cong \tau_{\leq 1}L\mathbf{SPr}_J(\mathscr{C})$, so this construction defines a method to construct *mapping stacks* between any two higher stacks, up to fibrant replacement.

4.2.4 Truncations

Another construction we would be rather interested in is that of *truncation*. A stack is to be interpreted as a sheaf of simplicial sets, so we may well be interested in considering the higher homotopies in every simplicial set up to some $n \geq 0$.

The way to achieve this is not so complicated. Indeed, we may define the truncated simplicial presheaves $\mathbf{SPr}_{\leq n}(\mathscr{C})$ to be functors $\mathscr{C}^{op} \to \mathbf{Fun}((\Delta^{\leq n})^{op}, \mathbf{Set})$, whose inclusion $\mathbf{SPr}_{\leq n}(\mathscr{C}) \to \mathbf{SPr}(\mathscr{C})$ has a left adjoint[3] $t_{\leq n}$. The composition of this with the inclusion will be the truncation functor we sought. There is an alternative method given by Toën and Vezzosi in [60, pg. 37-38] where one takes a left Bousfield localization of $\mathbf{SPr}_J(\mathscr{C})$ around n-truncated objects, which is then shown to be equivalent.

Our functor $t_{\leq n}$ clearly coincides with π_0^{ps} should we have $n = 0$, in fact sending stacks to sheaves in this case and giving us regular stacks if $n = 1$ [57, pg. 11]. This helps justify the universal construction of sheaves in terms of hypercovers at the beginning of this section.

4.2.5 Constructing Higher Stacks

It is somewhat unfortunate how technically involved our current definition for a higher stack still seems to be. Before we leave the foundations of higher stack theory behind, we should try to rein in the complex constructions we have developed and identify some more feasible methods to generate useful higher stacks. Toën suggests one such technique in [57, pg. 12-13].

[3]In [57, pg. 11], Toën says this should be a right adjoint. I believe this is a typo, as it contradicts what he says about π_0 both directly above and below it.

For a general Grothendieck site $\mathscr{C}$, we want some way to construct higher stacks, or simplicial presheaves satisfying a suitable descent condition. These should be representative of a moduli problem just like how ordinary stacks were; each simplicial set should be a set of families with weak equivalences between them. One possible starting point is to pick a so-called *presheaf of model categories on* $\mathscr{C}$, a functor $M : \mathscr{C}^{op} \to \mathbf{Cat}$ whose images of objects $M(X)$ are model categories and pullback morphisms are left Quillen functors. This lets us explicitly choose the equivalences between families that we want via the weak equvialences in each model category, which must be preserved by the left Quillen functors. Moreover, as we have seen in Section 3.5.5, we can usually expect a classifying space of objects to be the nerve of weak equivalences in a model category, so this is a reasonably general approach to producing higher stacks.

From this, we may construct a prestack $F \in \hat{\mathscr{C}}$ by setting

$$F(X) := |W^c_{M(X)}|$$

ie. the nerve of the subcategory of weak equivalences in $M(X)^c$, the cofibrant subcategory of $M(X)$. We restrict to cofibrant replacements to ensure weak equivalences are all preserved, though of course any model category's nerve of equivalences is equivalent to the nerve of acyclic cofibrations, owing to the same fact holding for localizations by Proposition 3.30 and the correspondence with nerves of equivalences. Hence, we have correctly defined $F(X)$ to be the classifying space of objects in $M(X)$ with the correct pullback morphisms.

What can we do now to enforce a descent condition, like hyperdescent? We will try to define a condition comparable to what we have seen for sheaves, where we demanded that matching families have a unique amalgamation. Consider a hypercover $U_* \to X$ in $\mathscr{C}$. We may construct the cosimplicial object $\overline{n} \mapsto M(U_n)$, where we have committed the abuse of notation that U_n is a formal disjoint union of objects and $M(U_n)$ is a product thereof. We may then define the category $\mathbf{Sect}(U_*, M)$ of *global sections*, whose objects are families of objects $x_n \in M(U_n)$ for all n and morphisms $\phi_u : u^*(x_m) \to x_n$ in $M(U_n)$ for every simplicial map $u : U_n \to U_m$, making the diagram

$$\begin{array}{ccc} u^*v^*(x_k) & \xrightarrow{u^*(\phi_u)} & u^*(x_j) \\ {\scriptstyle 1_{u^*v^*(x_k)}}\downarrow & & \downarrow{\scriptstyle \phi_u} \\ (v \circ u)^*(x_k) & \xrightarrow{\phi_{v \circ u}} & x_i \end{array}$$

commute [61, pg. 211]. We may define a model structure on $\mathbf{Sect}(U_*, M)$ where equivalences and fibrations are levelwise [57, pg. 13]. With this in mind, we have a natural functor

$$\phi : M(X) \to \mathbf{Sect}(U_*, M)$$

sending any x to the family of pullbacks $u^*(x)$ for all maps $U_n \to X$, together with identity functions for all ϕ_u. This functor preserves equivalences between cofibrant objects [61, pg. 213], which means it has a total left derived functor $\mathbb{L}\phi$ by Proposition 3.17. Furthermore, it has a right adjoint $\mathbb{R}\psi$ defined by taking a homotopy limit [61, pg. 213]; after all, this is just the constant diagram functor where our diagram is defined by the hypercover $U_* \to X$.

We will say that M satisfies *homotopical descent* if $\mathbb{L}\phi$ is fully faithful and its essential image is the collection of all objects $x_* \in \mathbf{Ho}(\mathbf{Sect}(U_*, M))$ where for any $u : U_n \to U_m$ the induced morphism $\mathbb{L}(u^*)(x_m) \to x_n$ by ϕ_u is an isomorphism in $\mathbf{Ho}(M(U_n))$ [57, pg. 13]. It is proven in [61, pg. 213-214] that if M satisfies homotopical descent then it must also satisfy hyperdescent. This suggests a rather natural way to produce higher stacks, as presheaves of model categories which satisfy homotopical descent seem to be somewhat closer to our prior experience with stacks and sheaves. Indeed, the resemblance to the sheaf condition is almost uncanny, weakened however to a homotopical context as we should expect from working with weak equivalences. Toën admits in [57] that this construction is the most powerful way he was aware of to construct higher stacks at the time of writing.

4.3 Higher Artin Stacks

If our concept of a higher stack is worth its salt, it should bring with it a sensible generalization of Artin stacks; the properties of Artin stacks were so well-suited to moduli theory that they should find a place in this wider context, where equivalences are not necessarily invertible. Toën and Vezzosi in [57] and [61] develop such a theory, which we follow here.

We begin by defining the Segal category $\mathbf{St}(k) := \mathbf{St}(\mathbf{Aff}_k)$ for any commutative ring k, where $\mathbf{Aff}_k$ is the site of affine schemes over $\mathbf{Spec}(k)$ with the étale topology[4]. The elements of $\mathbf{St}(k)$ will simply be called *k-stacks*. Since the étale topology is

[4] Toën in [57] uses the fpqc topology. However, Theorem 4.2 relies on all coverings being smooth in the full proof in [61, pg. 82], so I elect to replace it with étale as done in [61, pg. 123].

subcanonical [53, Tag 03PF], we have a fully faithful morphism $\mathbf{Aff}_k \to \mathbf{St}(k)$ factoring through the category of sheaves. Hence, we will choose to identify $\mathbf{Aff}_k$ with its essential image in $\mathbf{St}(k)$.

The elements of $\mathbf{St}(k)$ may be regarded, by our previous analysis, as simplicial presheaves

$$F : (\mathbf{Aff}_k)^{op} \to \mathbf{sSet}$$

such that for any étale hypercover of affine k-schemes $U_* \to X$, the induced morphism

$$F(X) \to \mathbb{R}\mathrm{lim}_{\overline{n}\in\Delta} F(U_n)$$

is an equivalence, where we have again made the abuse of notation $F(U_n)$ to mean $\prod_i F(u_i^n)$, where $U_n = \coprod_i u_i^n$.

We work with the Segal category $\mathbf{St}(k)$ henceforth, using fibre products $- \times_{-} -$ to mean the lifting of the homotopy fibre product in $\mathbf{Ho}(\mathbf{St}(k))$ to the Segal case. If we have need for the model category underlying it, we will write $\mathrm{St}(k)$ instead.

What is our intention for a higher Artin stack? We could quite easily extend the notion by simply generalizing representability, atlases and diagonals to our new notion of higher stack. However, there are some rather significant constructions we would be incapable of handling if we restricted ourselves to this point of view. One such case arises from considering groupoid actions in the category of Artin stacks. If we wanted this to form a stack of some kind, higher stacks seem somewhat immediately necessary, but forcing it into the definition of generalized Artin stack we have just imagined would be a difficult task.

Instead, we seek an inductive definition for a higher Artin stack that captures this case as well, letting us effectively study moduli problems of Artin stacks, moduli problems of these moduli stacks, *ad infinitum.* Some higher stacks will not take this form directly, instead consisting of a limit of inductively higher quotients of such higher stacks. This is a somewhat natural generalization of the research Alper, Hall and Rydin completed in [2] to show how Artin stacks are often locally quotient stacks; higher stacks should in general be locally quotients of schemes, then quotients of these stacks and so on.

As done in [61, pg. 76] and [57, pg. 14-15], the concept of a higher stack is defined somewhat differently, which we will prove to be equivalent to our motivation later on.

Definition 4.12. [57, pg. 14-15] [61, pg. 76-77] An *n-Artin stack* in $\mathbf{St}(k)$ for some commutative ring k is defined inductively as follows:

- A (-1)-Artin stack is an affine k-scheme. A morphism $f : F \to G$ between k-stacks is (-1)-*representable*, or *affine*, if for any affine k-scheme X and morphism $X \to G$ the pullback $F \times_G X$ is a (-1)-Artin stack. Furthermore, a (-1)-representable morphism is (-1)-*smooth* if for any affine k-scheme X and morphism $X \to G$, the pullback morphism

$$F \times_G X \to X$$

is a smooth morphism between affine k-schemes.

- For any $n \geq 0$, we will assume $(n-1)$-Artin stacks, $(n-1)$-representable morphisms and $(n-1)$-smooth morphisms are defined.
  - An n-*atlas* for a k-stack F is a set of morphisms $\{U_i \to F\}_{i \in I}$ such that each U_i is an affine k-scheme, each morphism $U_i \to F$ is $(n-1)$-smooth and the morphism
  $$\coprod_{i \in I} U_i \to F$$
  is an étale-covering.
  - A k-stack F is an n-*Artin stack* if the diagonal morphism $F \to F \times F$ is $(n-1)$-representable and F admits an n-atlas.
  - A morphism of k-stacks $F \to G$ is n-*representable* if for any affine k-scheme X and morphism $X \to G$, the pullback $F \times_G X$ is an n-Artin stack.
  - A morphism of k-stacks $F \to G$ is n-*smooth* if for any affine k-scheme X and morphism $X \to G$, there exists an n-atlas $\{U_i\}$ of $F \times_G X$ such that the composite morphisms $U_i \to X$ are all smooth morphisms of affine k-schemes.

A k-stack F which is an n-Artin stack for some n is also called an *Artin stack*, or *higher Artin stack* if we wish to distinguish from the 1-stack case. If F is equivalent to an n-truncated k-stack, we will call F an *Artin n-stack*. Similarly, a morphism $f : F \to G$ between k-stacks is *representable* if it is n-representable for some n.

Some important facts proven in [61, pg. 77] are that $(n-1)$-representable and $(n-1)$-smooth morphisms are also n-representable and n-smooth, respectively. Furthermore, both of these classes of morphisms contain all isomorphisms and are stable under homotopy pullback and composition. This means in particular that any $(n-1)$-Artin stack is also an n-Artin stack. It can also be shown that the full subcategory of

n-Artin stacks is closed under homotopy pullbacks and closed under disjoint unions if $n \geq 0$ [61, pg. 79].

As noted in [57], there is of course a discrepancy between the notions of n-Artin stack and Artin n-stack that the reader must be cautious of; the former is part of an inductive definition, while the latter is a truncation of a certain dimension. Indeed, a general scheme is certainly an Artin 0-stack, though is only a 1-Artin stack, lest its diagonal always be affine [57, pg. 15].

4.3.1 Quotient Stacks

Before we begin this section, we will need a generalization of a Segal groupoid to one over any model category:

Definition 4.13. [61, pg. 64] Given a model category $\mathscr{M}$, a *Segal groupoid object in* $\mathscr{M}$ is a simplicial object

$$X_* : \Delta^{op} \to \mathscr{M}$$

such that the following conditions are satisfied:

1. For every $n > 0$, the natural morphism

$$\prod_{0 \leq i < n} (\sigma_i)^* : X_n \to X_1 \times^h_{X_0} X_1 \times^h_{X_0} \cdots \times^h_{X_0} X_1$$

   is an isomorphism in $\mathbf{Ho}(\mathscr{M})$, where $\sigma_i = \langle i, i+1 \rangle : \overline{1} \to \overline{n}$.

2. The morphism

$$(d^0)^* \times (d^1)^* : X_2 \to X_1 \times^h_{(d^0)^*, X_0, (d^0)^*} X_1$$

   is an isomorphism in $\mathbf{Ho}(\mathscr{M})$.

We will try to motivate why this should be considered derivative of standard Segal groupoids in any way. Henceforth, we will drop the $(-)^*$ honorific on pullback morphisms, leaving this implicit. For a Segal groupoid object X_* of $\mathscr{M}$, consider the inverse in $\mathbf{Ho}(\mathscr{M})$ of the equivalence

$$\sigma_0 \times \sigma_1 : X_2 \to X_1 \times^h_{X_0} X_1$$

composed with $d^1 : X_2 \to X_1$. As with Segal categories, we get a well-defined *composition morphism*

$$\mu : X_1 \times^h_{X_0} X_1 \to X_1$$

in $\mathbf{Ho}(\mathscr{M})$ [61, pg. 80]. Similarly, composing the inverse in $\mathbf{Ho}(\mathscr{M})$ of the equivalence

$$d^0 \times d^1 : X_2 \to X_1 \times^h_{d^0, X_0, d^0} X_1$$

and composing on either side with the two morphisms

$$1_{X_1} \times^h s^0 : X_1 \to X_1 \times^h_{d_0, X_0, d_0} X_1 \qquad d^2 : X_2 \to X_1$$

gives us an isomorphism in $\mathbf{Ho}(\mathscr{M})$ $i : X_1 \to X_1$ which we dub *inversion* [61, pg. 80]. Note that that $d^1 \circ i = d^0$ for $d^1, d^0 : X_1 \to X_0$, meaning they are identical up to isomorphisms in $\mathbf{Ho}(\mathscr{M})$ [61, pg. 80].

We will now immediately repurpose this construction to formalize the groupoid actions on Artin stacks we alluded to previously.

Definition 4.14. [61, pg. 81] A Segal groupoid object X_* in $\mathbf{Ho}(\mathbf{SPr}(\mathbf{Aff}_k))$[5] is an *n-smooth Segal groupoid* if X_0 and X_1 are each disjoint unions of n-Artin stacks and the source morphism $d^0 : X_1 \to X_0$ is n-smooth.

Note that declaring the morphism d^0 to be n-smooth is valid, as taking disjoint unions like this can only give small disjoint unions, so X_0 and X_1 as above are in fact Artin stacks. The reason we need to go through this trouble is because (-1)-Artin stacks are not closed under disjoint union in general; such a disjoint union can only be guaranteed to be a 0-Artin stack.

The fact that n-geometric stacks are stable under homotopy pullbacks implies for any n-smooth Segal groupoid X_* that X_i is a disjoint union of n-Artin stacks for all i [61, pg. 81]. Furthermore, because the two morphisms $d^0, d^1 : X_1 \to X_0$ are equal up to isomorphism in $\mathbf{Ho}(\mathbf{SPr}(\mathbf{Aff}_k))$ [61, pg. 81] we see that for any n-smooth Segal groupoid X_*, all face maps $X_i \to X_{i-1}$ are n-smooth [61, pg. 81].

Theorem 4.2. [61, pg. 81] Let $F \in \mathbf{Ho}(\mathbf{SPr}(\mathbf{Aff}_k))$ be a stack and $n \geq 0$. The following conditions are equivalent:

1. F is n-Artin.

2. There is an $(n-1)$-smooth Segal groupoid object X_* in $\mathbf{SPr}(\mathbf{Aff}_k)$ such that X_0 is a disjoint union of representable stacks and there is an induced isomorphism in $\mathbf{Ho}(\mathbf{SPr}(\mathbf{Aff}_k))$

$$F \simeq |X_*| := \mathbb{L}\mathrm{colim}_{\overline{n} \in \Delta^{op}} X_n$$

[5] In [61], the category written here is in fact $\mathbf{Ho}(\mathbf{SPr}(\mathrm{St}(k)))$. I believe this to have been an error; such a category seems out of place and no proofs rely on it.

3. There is an $(n-1)$-smooth Segal groupoid X_* in $\mathbf{SPr}(\mathbf{Aff}_k)$ and there is an induced isomorphism in $\mathbf{Ho}(\mathbf{SPr}(\mathbf{Aff}_k))$

$$F \simeq |X_*| := \mathbb{L}\mathrm{colim}_{\overline{n}\in\Delta^{op}} X_n$$

We will call the colimits in conditions 2 and 3 above *quotient stacks* of the corresponding $(n-1)$-smooth Segal groupoid [61, pg. 81]. Proving this theorem directly goes somewhat beyond the scope of this dissertation, though at least the implication $2 \Rightarrow 3$ is immediate. To prove $1 \Rightarrow 2$, one essentially considers the induced morphism

$$p : X_0 := \coprod_i U_i \to F$$

and defines a simplicial object X_* to be the nerve of p. Toën and Vezzosi prove in [61, pg. 81] that, up to equivalent replacements of p, this is the desired Segal groupoid object.

An attractive consequence of this theorem is that the complex correspondence between quotient stacks and Artin stacks we briefly noted before is now simple and all-encompassing; every higher Artin stack is now a higher quotient stack. Furthermore, constructing quotients of Artin stacks is now rather straightforward, meaning a whole plethora of higher Artin stacks become available when we consider group actions on stacks.

4.3.2 Properties

It would be in our best interest to build up a toolkit of constructions and attributes of higher Artin stacks to enrich their applicability and ease of use. Perhaps the most important of these would be to identify how to extend properties of morphisms of schemes to ones of higher Artin stacks, much like we did for normal Artin stacks.

Definition 4.15. [61, pg. 87] Let P be a stable property of morphisms in $\mathbf{Aff}_k$ with the étale topology. A morphism of higher Artin stacks $f : F \to G$ *has property* P if it is n-representable for some n and if for any representable stack X and morphism $X \to G$ there exists an n-atlas $\{U_i\}$ of $F \times_G X$ such that each morphism $U_i \to X$ between representable stacks has property P.

This is just an extension of our definition for n-smoothness, or smoothness more generally. A wide range of properties now become available to us. For instance, a higher Artin stack F may be called *quasi-compact* if there is a surjective morphism of stacks $X \to F$ with X representable [57, pg. 15]. A morphism $f : F \to G$

between higher Artin stacks is then quasi-compact if for any affine X and morphism $X \to G$ the stack $F \times_G X$ is quasi-compact. We will say a 0-Artin stack is *strongly quasi-compact* if it is quasi-compact and by induction on n say an n-Artin stack is strongly quasi-compact if it is quasi-compact and the diagonal $F \to F \times F$ is strongly quasi-compact. In turn, a morphism $F \to G$ between n-Artin stacks is strongly quasi-compact if the fibre product with any morphism from an affine k-scheme is strongly quasi-compact, as with quasi-compactness [57, pg. 15].

Another property we may define is that of an immersion. Toën defines in [57, pg. 15] a morphism of higher Artin stacks $F \to G$ to be an *open immersion* if for any affine k-scheme X and morphism $X \to G$, the stack $F \times_G X$ is a scheme where the induced morphism $F \times_G X \to X$ is an open immersion. The same may be done to define a closed immersion.

Toën also declares a higher Artin stack F to be *locally of finite presentation* if it has a smooth atlas $U \to F$ such that the scheme U is of locally finite presentation, which when coupled with quasi-compactness makes F *strongly of finite presentation* [57, pg. 15]. One could imagine producing an ever-growing list of such properties now that properties of morphisms between affine k-schemes have been naturally extended to the realm of higher Artin stacks, a number of which are explained by Toën in [57, pg. 15-20].

One aspect of higher Artin stacks not derivative of any classical counterpart is that of *homotopy groups.* Given an affine k-scheme X, higher Artin stack F and morphism $x : X \to F$, Toën defines in [57, pg. 17] the *loop stack at x* as

$$\Omega_x F := X \times_F X$$

which, as we know, is a higher Artin stack over X. Combined with this, we notice that an induced morphism

$$\Omega_x F \times_X \Omega_x F \simeq X \times_F X \times_F X \to X \times_F X \simeq \Omega_x F$$

makes $\Omega_x F$ a group object in $\mathbf{St}(k)/X$ [57, pg. 17]. We may repeat this construction inductively to define

$$\Omega_x^n F := \Omega_x(\Omega_x^{n-1} F)$$

which is inductively a group stack over X itself [57, pg. 17]. This finally gives rise to the *n^{th} homotopy sheaf of F at x*

$$\pi_n(F, x) := \pi_0^{\text{ét}}(\Omega_x^n F)$$

which is of course a sheaf of groups, noted by Toën to be abelian for $n \geq 1$ [57, pg. 17] and thus coinciding somewhat perfectly with the classical homotopy groups we are accustomed to. It is proven in [61, pg. 131] that an Artin n-stack will have trivial sheaves π_m for $m > n$.

As an aside that we will make no attempt to prove here, Toën proves in [56, pg. 14] that if $X = \mathrm{Spec}(k)$ and F is strongly of finite presentation then $\pi_n(F, x)$ is in fact representable by a finitely presentable group scheme over $\mathrm{Spec}(k)$. This group scheme represents the n-automorphisms of the point x in F [57, pg. 17].

With homotopy groups come Eilenberg-MacLane spaces. In [56, pg. 8], for a sheaf of abelian groups G, Toën defines a simplicial presheaf

$$\begin{aligned} K(G, n) : (\mathbf{Aff}_k)^{op} &\to \mathbf{sSet} \\ A &\mapsto \mathrm{Sing}(K(G(A), n)) \end{aligned}$$

which is then stackified. It may then be shown, as is standard, that for any affine k-scheme X we have functorial bijections

$$\pi_0(\mathbf{St}(k)_{(X, K(G,n))}) \cong H^n_{\text{ét}}(X, G)$$

[56, pg. 9] [57, pg. 21]; Toën shows this for fppf and fpqc topologies, though this is simply because the site $\mathbf{Aff}_k$ he used had these topologies itself. It is now possible to define a cohomology on all higher Artin stacks F and sheaves of abelian groups G via

$$H^n(F, G) := \pi_0(\mathbf{St}(k)_{(F, K(G,n))})$$

[57, pg. 21].

A final fact that we will pay brief homage to is that for any objects F and G in $\mathbf{St}(k)$, the functor $\mathbf{St}(k) \to \mathrm{Top}$ defined by $H \mapsto \mathbf{St}(k)_{(H \times F, G)}$ is in fact representable by an object we will write as $\mathbf{Map}(F, G) \in \mathbf{St}(k)$ [57, pg. 24]. This is of course just the internal structure on higher stacks we have seen in a more abstract context previously. Toën notes however that if F is a smooth and proper scheme and G is an Artin n-stack locally of finite presentation, then $\mathbf{Map}(F, G)$ will itself be an Artin n-stack locally of finite presentation [57, pg. 24]. While this condition is sufficient, Toën notes more general cases where $\mathbf{Map}(F, G)$ is a higher Artin stack in [57, pg. 24], so it is not necessary. The identification of suitable such conditions is left as future work.

4.3.3 Examples

Finally, we are able to tackle some substantial examples of moduli problems where our new higher Artin stacks may be employed. Exploring these in complete detail would go somewhat beyond the scope of this dissertation, so we will present an overview of some that Toën describes in [57, pg. 22-23].

4.3.3.1 Perfect Complexes

The first case we consider, described by Toën in [57, pg. 22-23], is that of *perfect complexes* of R-modules for any commutative k-algebra R up to quasi-isomorphism. We provide a definition for the reader's convenience:

Definition 4.16. [53, Tag 0657] A chain complex of R-modules is called *perfect* if it is quasi-isomorphic to a bounded complex of finite projective R-modules.

We are interested in perfect complexes composed of flat R-modules. The category of these with quasi-isomorphisms as morphisms, denoted $\mathbf{Perf}(R)$, comes by definition with a well-defined functor for any morphism $R \to Q$ of k-modules

$$- \otimes_R Q : \mathbf{Perf}(R) \to \mathbf{Perf}(Q)$$

as noted in [57, pg. 22]. Of course, projectivity implies flatness, so this is really not such a harsh demand. Now blatantly assuming a Grothendieck universe foundation, we may consider the nerve of any such category and retrieve a simplicial presheaf

$$\begin{aligned} \mathbf{Perf} : (\mathbf{Aff}_k)^{op} &\to \mathbf{sSet} \\ \mathrm{Spec}(R) &\mapsto N(\mathbf{Perf}(R)) \end{aligned}$$

and sending morphisms to the induced base change functor. It is possible, using the techniques of homotopical descent, to prove that this is indeed a stack [57, pg. 22-23] in $\mathbf{St}(k)$. It is also known that $\pi_0(\mathbf{Perf}(\mathrm{Spec}(R)))$ bijects naturally to isomorphism classes of $D_{perf}(R)$, the perfect derived category of R. Furthermore, for any perfect complex $E \in \mathbf{Perf}(R)$, we have that $\pi_1(|\mathbf{Perf}(\mathrm{Spec}(R))|, E)$ is naturally isomorphic to the automorphism group of $E \in D_{perf}(R)$ and $\pi_i(|\mathbf{Perf}(\mathrm{Spec}(R))|, E)$ is naturally isomorphic to $\mathrm{Ext}^{1-i}(E, E)$ for $i > 1$ [57, pg. 23][6].

[6] I have assumed that this is what Toën means by the homotopy groups of a simplicial set; the direct definition of a simplicial homotopy group in [20, pg. 26] requires the simplicial set in question to be fibrant, in which case it agrees with what I have written by [20, pg. 65]. This is not clarified in [59] either, where it is proven more generally.

This stack is, of course, not truncated due to it ranging over perfect complexes of any size, meaning we cannot declare it an Artin n-stack for some n. However, all is not lost: such structure can be found locally instead. It is proven in [59, pg. 41] that the substacks $\mathbf{Perf}^{[a,b]}$ of complexes of amplitude $[a, b] \subset \mathbb{Z}$ are Artin $(b-a+1)$-stacks locally of finite presentation, which in turn implies **Perf** is in fact a union of growing Artin substacks along the inclusions $\mathbf{Perf}^{[a,b]} \hookrightarrow \mathbf{Perf}^{[a',b']}$ for $a' \leq a \leq b \leq b'$. We denote this property by declaring **Perf** to be *locally geometric* [57, pg. 23]. Hence, if we are working in any local area of the stack, we may presume we are in an Artin n-stack for some n.

4.3.3.2 Abelian Categories

Higher stack theory not only expands the scope of moduli problems we may tackle but also the information we may gleam from ones accessible to us already. As an example of this phenomenon, we consider a moduli problem of abelian categories. We may, for any commutative k-algebra R, define $R - Ab$ to be the category of abelian R-linear categories A equivalent to $B - mod$ for some associative R-algebra B that is projective and of finite type as an R-module [57, pg. 22]. With this, we set the morphisms to be R-linear equivalences. Note how our equivalences in this case are indeed invertible.

We may convert any morphism $R \to Q$ of commutative k-algebras into a base change functor $R - Ab \to Q - Ab$, giving us a simplicial presheaf if we are willing to yet again dive into a Grothendieck universe foundation

$$\begin{aligned} \mathbf{Ab} : (\mathbf{Aff}_k)^{op} &\to \mathbf{sSet} \\ \mathrm{Spec}(R) &\mapsto N(R - Ab) \end{aligned}$$

sending any affine variety to abelian categories defined over its ring of global sections [57, pg. 22]. As a matter of fact, this is *not* a stack [57, pg. 22]; we must stackify it first, giving us what we will now call **Ab** to replace the simplicial presheaf above.

The homotopy groups of this are in fact known, as described in [57, pg. 22]. It is the case that $\pi_0(\mathbf{Ab}(\mathrm{Spec}(R)))$ is just the equivalence classes of abelian R-linear categories in $R - Ab$ and for any $A \in \mathbf{Ab}(\mathrm{Spec}(R))$, the group $\pi_1(|\mathbf{Ab}(\mathrm{Spec}(R))|, A)$ is naturally isomorphic to the isomorphism classes of automorphisms of A. The group $\pi_2(|\mathbf{Ab}(\mathrm{Spec}(R))|, A)$ is then the automorphism group of 1_A and, strikingly, all higher homotopy groups are trivial[7]. One may then show **Ab** to be an Artin 2-stack locally of finite presentation over $\mathrm{Spec}(k)$ [57, pg. 22], a property that regular stack theory could not have shown us.

[7]The same assumptions about homotopy groups made in the prior example apply here.

Chapter 5

Conclusion

This dissertation began with a simple question:

> *Can we construct a mathematical object that makes explicit the geometric variations in a collection of spaces?*

It is in the pursuit of a truly satisfying answer that we have found a story of increasingly abstract and powerful constructions. The tale began with a collection of spaces with trivial relations to one another, solved quickly by the clear and simple approach of moduli spaces. With symmetries comes complexity, however; even matrices up to change of basis failed to meet the subtly unfair standards moduli space theory demands. We were forced to turn to sheaves as a neat generalization of a space, giving us stack theory once these symmetries were included.

Of course, understanding spaces up to equivalence is a somewhat loaded expression: the equivalences in question may not truly be isomorphisms, instead representing some preservation of structure that loses unimportant information in an irreversible manner. Examples like homotopy and quasi-isomorphism spring to mind at this stage. We noted classical stack theory to be nowhere near ready to meet the challenges of such a situation; a more nuanced understanding of the relationships between morphisms was in order. As is natural in the eyes of category theory, such structure must come from morphisms between morphisms, leading us into the *higher category theory* that would serve as the foundation for the more powerful notion of a *higher stack*. In the end, this construction would prove up to the challenge of solving our leading question in perhaps the most general setting imaginable.

In our case, this tool was fine-tuned particularly for the setting of algebraic geometry, giving us a notion of higher Artin stack to deal with moduli problems in this context. A second story thus unfolded in our discussions, one where we realized that the equivalences in a moduli problem are often best expressed by a group action. This highlighted a crucial thread connecting moduli spaces to categorical quotients and orbit spaces, a thread remaining unsevered under the strain of generalization to

Artin stacks and quotient stacks. In fact, we have seen it grow only tighter, eventually pulling its two ends together as higher Artin stacks were revealed to be nothing more than an inductive quotient of affine schemes themselves. Our long climb through the theory of higher categories has earned us a wonderful view of geometry below; from a point so high, what was once beyond the horizon now seems to lie in plain sight.

5.1 Future Work

Despite how far we have come, there remains still more work to be done. Toën dares to step even beyond the confines of higher Artin stack theory, developing *derived algebraic geometry* by changing our source category from $(\mathbf{Aff}_k)^{op}$, equivalently the category of commutative k-algebras $k-CAlg$, to the category $sk-CAlg$ of simplicial objects in $k-CAlg$ [57, pg. 32]. He argues this to be a setting where notions such as obstruction theory become natural [57, pg. 31], so it would be sensible to extend the discussion in this dissertation to that context. It is unfortunate that the theory involved is too vast to include and justify properly in this dissertation.

There are in fact multiple notions of a higher stack that should be correlated. Hirschowitz and Simpson in [26] develop a notion of stacks on Segal categories and Lurie in [30] produces his generalized notion of an ∞-*stack* using the abstract notion of an ∞-category. It would be in our best interest to compare and contrast these different formalisms in full detail, though explicitly defining these other approaches to higher stacks goes somewhat beyond the scope of this dissertation.

Some points need to be proven consistent. For instance, Bergner's model structure on Segal precategories and Hischowitz and Simpson's need a more thorough comparison to ensure that the theory of simplicial categories and Segal categories are consistent in a way that corresponds to the theory we have discussed elsewhere. Furthermore, it must be proven that the $\mathbb{R}$Hom's in **PrSeCat** are indeed equivalent to a suitable Segal category, as this is a point often used without evidence I was able to find.

The universal property of higher stacks we proposed needs to be justified fully; I was able to prove it partially, though the fact that the induced functor's image contains every possible functor satisfying the colimit condition on hypercovers is unclear. I believe that the theory of Kan extensions in enriched categories developed in [29, ch. 4] will be central to a more complete proof, particularly theorem 4.51 pertaining to the enriched Yoneda embedding.

Another significant point is that many of the higher stack constructions we have produced are not just done with a site $\mathscr{C}$ but with a *Segal site* or *simplicial site*, namely a Segal or simplicial category $\mathscr{C}$ with a topology on $\mathbf{Ho}(\mathscr{C})$ [62, pg. 10] [60, pg. 18]. This requires us to consider constructions like simplicial model categories that were mentioned briefly. Developing this in full was unnecessary and excessive for our purposes, though would be a requirement to fully prove all things like the hyperdescent characterization of stacks in $\mathbf{SPr}_J(\mathscr{C})$. The reader may rest easy knowing that the structures we have developed thus far are indeed special cases of these ones.

Finally, a few minor points deserve consideration. The ring structure on the cohomology theory of Artin stacks we developed needs clarification, should a sensible one exist. The higher path and cylinder objects we developed need to be refined into true simplicial and cosimplicial resolutions; there is no reason that such intuitive examples of resolutions should fail to satisfy all the necessary properties. As well as this, many of the formalisms of higher categories we have employed, such as Segal categories, should ideally be redesigned so as not to be sets themselves, as this unnecessarily restricts us to certain foundational systems should we wish to use them on general categories.

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