Simplicial Sets: The Convergence of Homotopy and Category Theory

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A Refresher on Homotopy Theory

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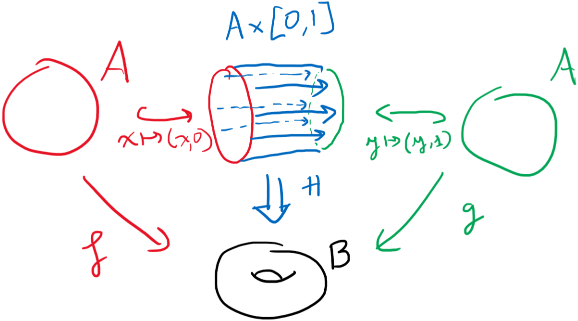
A **left homotopy** between maps $f, g : A \to B$ (write $f \sim g$):



Homotopy Equivalence

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A functor $F : \mathscr{C} \to \mathscr{D}$ is an *equivalence of categories* if there is a functor $G : \mathscr{D} \to \mathscr{C}$ such that $F \circ G \sim 1_{\mathscr{D}}$ and $G \circ F \sim 1_{\mathscr{C}}$.

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Identical definitions if we substitute **categories** for **spaces**, **functors** for **continuous maps** and **natural isomorphisms** for **homotopies**! How far does this analogy go...?

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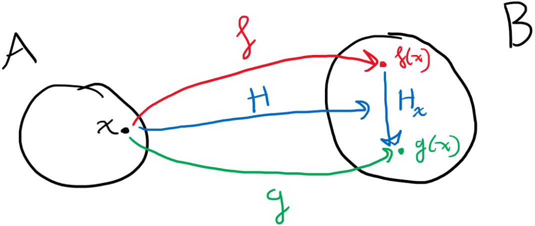
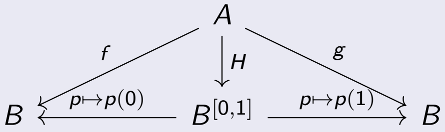


Right Homotopy

Definition 3

A *right homotopy* between continuous maps $f, g : A \to B$ is a continuous map $H : A \to B^{[0,1]}$ such that the diagram



commutes. We say f is *right homotopic* to g.

Proposition 1

Right homotopy forms an equivalence relation identical to left homotopy.

Path Groupoids

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Definition 4

The *path groupoid* of a topological space A is the category $\Pi_1(A)$ whose objects are points $x \in A$ and morphisms in $\mathbf{Hom}_{\Pi_1(A)}(x, y)$ are paths $p : [0, 1] \to A$ where $p(0) = x$ and $p(1) = y$, up to homotopies fixing x and y.

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- $\Pi_1(A)$ is a category for all A
- Π_1 is a functor **Top** $\to$ **Cat**
- Right homotopies in **Top** become natural isomorphisms in **Cat**, respecting composition and identities
- Homotopy equivalences are thus sent to equivalences of categories - $\Pi_1(A)$ is a homotopy invariant!

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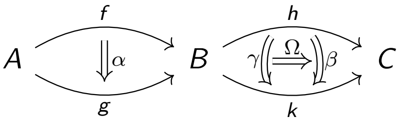
Want higher morphisms between morphisms:



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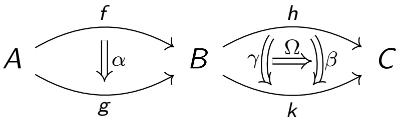

Say A, B, C are objects, f, g, h, k are 1-morphisms, α, β, γ are 2-morphisms, Ω is a 3-morphism.

Homotopy ∞-Groupoids

- For a space A, let $\Pi_\infty(A)$ be the higher category whose objects are points in A, morphisms are paths $[0,1] \to A$ and n-cells are in general maps $[0,1]^n \to A$

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QUESTION: What is a higher groupoid?

Simplicial Sets

- We will formalize higher groupoids with *simplicial sets*

Homotopy Theory

Simplicial Sets

Kan Complexes

Sing and $|\cdot|$

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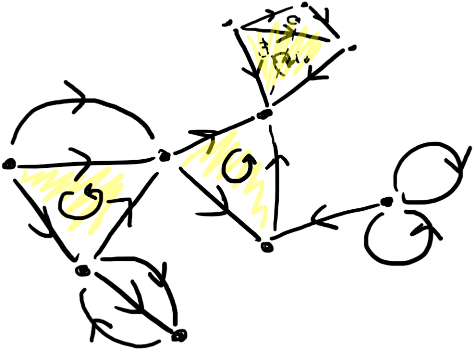

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- Starting to look like a contravariant functor from a 'category of simplices' to **Set**

The Simplicial Category

Definition 5

The *simplicial category* Δ is the category whose objects are totally ordered sets of the form

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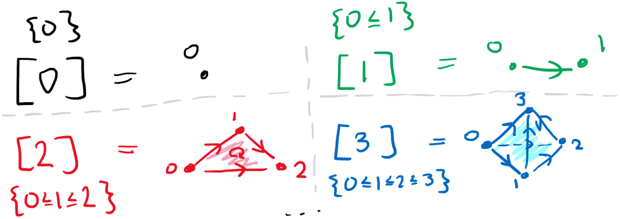

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Such a functor:

- Sends the n-simplex $[n]$ to the set of n-simplices

$$X_n := X([n])$$

- Sends the map $f : [n] \to [m]$ to a reverse map

$$f^* := X(f) : X_m \to X_n$$

extracting an n-simplex from each m-simplex as per f

Category of Simplicial Sets

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Important example of a simplicial set:

Definition 8

The n-simplex $\Delta^n \in \mathbf{sSet}$ is the simplicial set represented by $[n]$, ie. with sets

$$(\Delta^n)_k := \mathbf{Hom}_\Delta([k], [n])$$

for all k.

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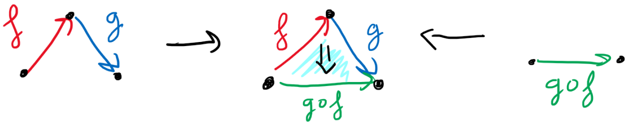

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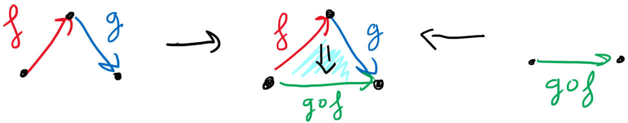

- The morphism $s^0 : [1] \to [0] \in \Delta$ induces a map $X_0 \to X_1$, sending $x \in X_0$ to its *identity map* $1_x \in X_1$

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- If every map in a quasicategory is 'invertible', we get a higher groupoid, or a **Kan complex**

Homotopies of Simplicial Sets

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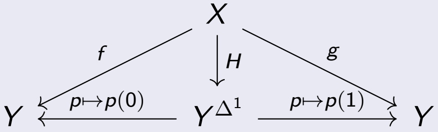
Definition 9

A *simplicial right homotopy* between maps of simplicial sets $f, g : X \to Y$ is a map $H : X \to Y^{\Delta^1}$ where the diagram



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The *topological n-simplex* $\Delta^n \in$ **Top** is the subspace of $\mathbb{R}^{n+1}$

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Definition 11

The *singular set* functor $\mathbf{Sing} : \mathbf{Top} \to \mathbf{sSet}$ sends a topological space A to the simplicial set $\mathbf{Sing}(A)$, such that

$$\mathbf{Sing}(A)_n := \mathbf{Hom}_{\mathbf{Top}}(\Delta^n_{top}, A)$$

for all n, with simplicial maps defined by precomposition.

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Sing has a right adjoint $|\cdot| : \mathbf{sSet} \to \mathbf{Top}$, sending a simplicial set X to its *geometric realization* $|X| \in \mathbf{Top}$.

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This is not the entire story - need *model categories* for that! (See a future talk...)

Questions?

Theorem 12

I will remember every name in this Zoom call that doesn't ask a question