Introduction to Algebraic Topology

Jack Romo

University of York

jr1161@york.ac.uk

June 2019

Introduction

- Based on lecture notes from the Oxford course *Topology and Groups*, taught by Prof. Marc Lackenby
- Assumes familiarity with basic topology, but everything we need will be re-proven properly!
- Group theory recommended but not essential, crash course provided at the beginning

Contents

Preliminary Group Theory

Definition 1 (Group)

A *group* is a pair $G = \langle S, * \rangle$, where S is a set and $* : S \times S \to S$ is a binary operation, such that

1. $\exists 1_G \in S$ such that $g * 1_G = g$ for $g \in S$;
2. $(g * h) * k = g * (h * k)$ for $g, h, k \in S$;
3. For $g \in S$, $\exists g^{-1} \in S$ such that $g * g^{-1} = 1_G$.

We often write $g \in G$ to mean $g \in S$, and treat G itself as a set. We also often contract $g * h$ to gh.

Basic Laws

Proposition 1

For any group G and $g, h \in G$,

$$g * 1_G = g = 1_G * g \tag{1}$$

$$g * g^{-1} = 1_G = g^{-1} * g \tag{2}$$

$$(g * h)^{-1} = h^{-1} * g^{-1} \tag{3}$$

$$g^{-1}, 1_G \text{ are unique.} \tag{4}$$

Group Homomorphisms

Definition 2 (Group Homomorphisms)

For two groups G, H, a *homomorphism* $\theta : G \to H$ is a function such that for all $g_1, g_2 \in G$,

$$\theta(g_1 * g_2) = \theta(g_1) * \theta(g_2)$$

where the latter binary operation is that of H.
An *isomorphism* is a bijective homomorphism.

Group Homomorphisms

Definition 2 (Group Homomorphisms)

For two groups G, H, a *homomorphism* $\theta : G \to H$ is a function such that for all $g_1, g_2 \in G$,

$$\theta(g_1 * g_2) = \theta(g_1) * \theta(g_2)$$

where the latter binary operation is that of H.
An *isomorphism* is a bijective homomorphism.

Proposition 2

For any groups G, H and a homomorphism $\theta : G \to H$,

$$\theta(1_G) = 1_H \qquad (5)$$

$$\theta(g^{-1}) = \theta(g)^{-1}. \qquad (6)$$

Subgroups

Definition 3 (Subgroup)

Given a group G, a *subgroup* S of G is a subset of G that is a group itself, with the same binary operation restricted to its elements. We write $S \leq G$ in this case.

Necessarily, we have $1_G \in S$ and for $a, b \in S$, $ab \in S$ and $a^{-1} \in S$. These criteria are a sufficient test for a subgroup.

Subgroups

Definition 3 (Subgroup)

Given a group G, a *subgroup* S of G is a subset of G that is a group itself, with the same binary operation restricted to its elements. We write $S \leq G$ in this case.

Necessarily, we have $1_G \in S$ and for $a, b \in S$, $ab \in S$ and $a^{-1} \in S$. These criteria are a sufficient test for a subgroup.

Definition 4 (Normal Subgroup)

A subgroup $N \leq G$ is a *normal subgroup*, written $N \triangleleft G$, iff for all $g \in G, n \in N$, we have $g^{-1}ng \in N$.

Quotient Groups

Definition 5 (Left Cosets)

Given a subset $S \subseteq G$ of a group G and $g \in G$, define the *left coset* gS as

$$gS = \{gs \mid s \in S\}$$

Quotient Groups

Definition 5 (Left Cosets)

Given a subset $S \subseteq G$ of a group G and $g \in G$, define the *left coset* gS as

$$gS = \{gs \mid s \in S\}$$

Definition 6 (Quotient Groups)

Given a normal subgroup $N \lhd G$, the *quotient group* G/N is the group whose elements are the left cosets gN for $g \in G$ and

$$(g_1 N) * (g_2 N) = \{ab \mid a \in g_1 N, b \in g_2 N\}$$

It turns out that if N is normal, $(g_1 N) * (g_2 N) = (g_1 * g_2)N$, a requirement for G/N to satisfy the axioms of a group.

Kernel and Image

Definition 7 (Kernel)

Given a homomorphism $\theta : G \to H$, the *kernel* $\ker \theta \subseteq G$ is defined as

$$\ker \theta = \theta^{-1}(1_H).$$

Definition 8 (Image)

Given $\theta : G \to H$ as above, the *image* $\mathrm{Im}\, \theta \subseteq H$ is defined as

$$\mathrm{Im}\, \theta = \theta(G).$$

Kernel and Image

Proposition 3

For a homomorphism $\theta : G \to H$, $\mathrm{Im}\,\theta \leq H$ *and* $\ker\theta \lhd G$.

Proposition 4

A homomorphism $\theta : G \to H$ *is injective iff* $\ker\theta = \{1_G\}$.

Generating Sets

Definition 9 (Generating Set)

A subset of a group $S \subseteq G$ is said to be a *generating set* iff every $g \in G$ is a product of elements of S and their inverses.

Generating Sets

Definition 9 (Generating Set)

A subset of a group $S \subseteq G$ is said to be a *generating set* iff every $g \in G$ is a product of elements of S and their inverses.

For instance, a generating set of $\langle \mathbb{Z}, + \rangle$ is $\{1\}$.
Another generating set is $\{2, 3\}$.

Generating Sets

Definition 9 (Generating Set)

A subset of a group $S \subseteq G$ is said to be a *generating set* iff every $g \in G$ is a product of elements of S and their inverses.

For instance, a generating set of $\langle \mathbb{Z}, + \rangle$ is $\{1\}$.
Another generating set is $\{2, 3\}$.
$\{2, 6\}$ is NOT a generating set.

Graphs

Definition 10 (Graph)

A *graph* $\Gamma = \langle V, E, \delta \rangle$ consists of a set of vertices V, a set of edges E and $\delta : E \to \mathcal{P}(V)$ which sends each edge to a subset of V with 1 or 2 elements. We call $\delta(e)$ the endpoints of e.

Graphs

Definition 10 (Graph)

A *graph* $\Gamma = \langle V, E, \delta \rangle$ consists of a set of vertices V, a set of edges E and $\delta : E \to \mathcal{P}(V)$ which sends each edge to a subset of V with 1 or 2 elements. We call $\delta(e)$ the endpoints of e.

Definition 11 (Orientation)

An *oriented graph* is a graph Γ together with functions $\iota : E \to V$ and $\tau : E \to V$ such that $\delta(e) = \{\iota(e), \tau(e)\}$ for all $e \in E$. We call ι and τ the source and target functions.

An *oriented graph* is a graph Γ together with an orientation.

Cayley Graphs

Definition 12 (Cayley Graph)

For a group G and a generating set $S \subseteq G$, the *Cayley graph* is an oriented graph with vertex set G and edge set $G \times S$, such that

$$\iota : \langle g, s \rangle \mapsto g$$
$$\tau : \langle g, s \rangle \mapsto gs$$

for all $g \in G$, $s \in S$.

Cayley Graphs

Definition 12 (Cayley Graph)

For a group G and a generating set $S \subseteq G$, the *Cayley graph* is an oriented graph with vertex set G and edge set $G \times S$, such that

$$\iota : \langle g, s \rangle \mapsto g$$
$$\tau : \langle g, s \rangle \mapsto gs$$

for all $g \in G$, $s \in S$.

Proposition 5

Any two points in a Cayley graph can be joined by a path.

Constructing Spaces

- Turns out many spaces can be constructed from simpler, finite ones
- Will define some useful methods to construct spaces here, in particular *simplicial complexes* and *cell complexes*

Simplices

Definition 13 (Simplex)

The *standard n-simplex* is the set

$$\Delta^n = \left\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \;\middle|\; x_i \geq 0 \,\forall\, i, \sum_{i=0}^{n} x_n = 1 \right\}$$

Simplices

Definition 14 (Vertices and Faces)

The *vertices* $V(\Delta^n)$ are all the elements of Δ^n where $x_i = 1$ for some $0 \leq i \leq n$.

Given a non-empty subset $A \subseteq \{0, \ldots, n\}$, a *face* of Δ^n is the subset

$$\{(x_0, \ldots, x_n) \in \Delta^n \mid x_i = 0 \, \forall \, i \notin A\}$$

Simplices

Definition 14 (Vertices and Faces)

The *vertices* $V(\Delta^n)$ are all the elements of Δ^n where $x_i = 1$ for some $0 \leq i \leq n$.

Given a non-empty subset $A \subseteq \{0, \ldots, n\}$, a *face* of Δ^n is the subset

$$\{(x_0, \ldots, x_n) \in \Delta^n \mid x_i = 0 \, \forall \, i \notin A\}$$

Definition 15 (Inside)

The *inside* of a simplex Δ^n is the set

$$\text{inside}(\Delta^n) = \{(x_0, \ldots, x_n) \in \Delta^n \mid x_i > 0 \, \forall \, i\}$$

Simplices

Definition 16 (Affine Extension)

For $f : V(\Delta^n) \to \mathbb{R}^m$, the unique linear extension of f to $\mathbb{R}^{n+1}$ then restricted to Δ^n is the *affine extension of* f.

Simplices

Definition 16 (Affine Extension)

For $f : V(\Delta^n) \to \mathbb{R}^m$, the unique linear extension of f to $\mathbb{R}^{n+1}$ then restricted to Δ^n is the *affine extension of f*.

Definition 17 (Face Inclusion)

A *face inclusion* of a standard m-simplex into a standard n-simplex, for $m < n$, is the affine extension of an injection $V(\Delta^m) \to V(\Delta^n)$.

Abstract Simplicial Complexes

Definition 18 (Abstract Simplicial Complex)

An *abstract simplicial complex* is a pair $\langle V, \Sigma \rangle$, where V is a set of 'vertices' and Σ is a set of finite subsets of V such that

1. for each $v \in V$, $\{v\} \in \Sigma$;
2. if $\sigma \in \Sigma$, so is every nonempty subset of σ.

Say that $\langle V, \Sigma \rangle$ is finite if V is finite.
We see the sets $\sigma \in \Sigma$ as sets of vertices for $(|\sigma| - 1)$-simplices.

Abstract Simplicial Complexes

Definition 19 (Topological Realization)

The *topological realization* $|K|$ of an abstract simplicial complex $K = \langle V, \Sigma \rangle$ is the space obtained by:

1. For every $\sigma \in \Sigma$, taking a copy of the standard $(|\sigma| - 1)$-simplex called Δ_σ, whose vertices are labelled with elements of σ;
2. For every $\sigma \subset \tau \in \Sigma$, identifying Δ_σ with a subset of Δ_τ by the face inclusion f where all $v \in V(\Delta_\sigma)$ and $f(v) \in V(\Delta_\tau)$ share the same label.

Note $|K|$ is a quotient space of the disjoint union of the simplicial realizations of each $\sigma \in \Sigma$.

Abstract Simplicial Complexes

- Note any point $x \in |K|$ is within some n-simplex, and is a linear combination of the vertices
- So, if $V = \{w_0, \ldots, w_n\}$, we have

$$x = \sum_{i=0}^{n} \lambda_i w_i$$

for $\lambda_i \in [0,1]$, $\sum \lambda_i = 1$, with the understanding that $\lambda_i = 0$ if x is not in the respective simplex

- From now on, we say 'simplicial complex' to refer either to an abstract simplicial complex or its topological realization

Triangulations

Definition 20 (Triangulation)

A *triangulation* of a topological space X is a simplicial complex K together with a homeomorphism $h : |K| \to X$.

Examples: $I \times I$, the torus $\mathbb{T}^2$

Subcomplexes and Maps

Definition 21 (Subcomplex)

A *subcomplex* of a simplicial complex $\langle V, \Sigma \rangle$ is a simplicial complex $\langle V', \Sigma' \rangle$ such that $V' \subseteq V$, $\Sigma' \subseteq \Sigma$.

Subcomplexes and Maps

Definition 21 (Subcomplex)

A *subcomplex* of a simplicial complex $\langle V, \Sigma \rangle$ is a simplicial complex $\langle V', \Sigma' \rangle$ such that $V' \subseteq V$, $\Sigma' \subseteq \Sigma$.

Definition 22 (Simplicial Map)

A *simplicial map* between abstract simplicial complexes $\langle V_1, \Sigma_1 \rangle$ and $\langle V_2, \Sigma_2 \rangle$ is a function $f : V_1 \to V_2$ such that, for all $\sigma \in \Sigma_1$, $f(\sigma) \in \Sigma_2$.

A simplicial map is a *simplicial isomorphism* if it has a simplicial inverse.

This induces a natural continuous map $|f| : |K_1| \to |K_2|$ by affine extension of f. We also call this a simplicial map.

Subdivisions

Triangulations are not unique; indeed, we may 'refine' one in a natural way!

Definition 23 (Subdivision)

A *subdivision* of a simplicial complex K is a triangulation K', $h : |K'| \to |K|$ of $|K|$ such that, for any simplex σ' in K', $h(\sigma')$ is entirely contained in some simplex of $|K|$ and the restriction of h to σ' is affine.

Example: $(I \times I)_{(r)}$ for $r \in \mathbb{N}$. (A subdivision we will use often!)

Cell Complexes

Simplicial complexes are useful for finitary arguments but a bit awkward to use directly. Thankfully, there is an alternative!

Definition 24 (Attaching n-cells)

Let X be a space and $f : S^{n-1} \to X$ be continuous. Then the space *obtained by attaching an n-cell to X along f*, denoted $X \cup_f D^n$, is the quotient of the disjoint union $X \sqcup D^n$ such that the equivalence classes are $f^{-1}(\{x\}) \cup \{x\}$ for every $x \in X$.

NB: We consider $S^{n-1} \subset D^n$ to be the boundary of D^n above, where D^n is the n-dimensional closed disk.

Cell Complexes

Definition 25 (Cell Complex)

A *(finite) cell complex* is a space X decomposed as

$$K^0 \subset K^1 \subset \cdots \subset K^n = X$$

where

1. K^0 is a finite set of points, and
2. K^i is obtained from K^{i-1} by attaching a finite number of i-cells.

Any finite simplicial complex is clearly a finite cell complex; let each n-simplex be an n-cell.

Examples: The torus, finite graphs

Homotopy

- A major topological property we can explore algebraically
- We will redefine all that we need from the ground up
- A major result: the Simplicial Approximation Theorem - from continuous functions to simplicial maps

Homotopy

Let X and Y henceforth be topologoical spaces.

Definition 26 (Homotopy)

A *homotopy* between two continuous maps $f : X \to Y$, $g : X \to Y$ is a continuous map $H : X \times I \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$. We would then say f and g are *homotopic*, written $f \simeq g$ or $f \overset{H}{\simeq} g$.

Homotopy

Let X and Y henceforth be topologoical spaces.

Definition 26 (Homotopy)

A *homotopy* between two continuous maps $f : X \to Y$, $g : X \to Y$ is a continuous map $H : X \times I \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$. We would then say f and g are *homotopic*, written $f \simeq g$ or $f \overset{H}{\simeq} g$.

A standard homotopy is the *straight-line homotopy*, defined as

$$H(x, t) = (1 - t)f(x) + tg(x)$$

Homotopy as an Equivalence

Lemma 27 (Gluing Lemma)

If $\{C_1, \ldots, C_n\}$ is a finite covering of a space X by closed subsets and the restriction of $f : X \to Y$ to each C_i is continuous, then f is continuous.

Homotopy as an Equivalence

Lemma 27 (Gluing Lemma)

If $\{C_1, \ldots, C_n\}$ is a finite covering of a space X by closed subsets and the restriction of $f : X \to Y$ to each C_i is continuous, then f is continuous.

Lemma 28

Homotopy is an equivalence relation on $\mathcal{C}(X, Y)$, the set of continuous maps $X \to Y$.

Composition of Homotopies

Lemma 29

Consider the following continuous maps:

$$W \xrightarrow{f} X \underset{h}{\overset{g}{\rightrightarrows}} Y \xrightarrow{k} Z$$

Then $g \simeq h$ *implies* $gf \simeq hf$ *and* $kg \simeq kh$.

Homotopy Equivalence

Definition 30 (Homotopy Equivalence)

Two spaces X and Y are *homotopy equivalent*, written $X \simeq Y$, if and only if there exist maps

$$X \underset{g}{\overset{f}{\rightleftarrows}} Y$$

such that $gf \simeq \mathrm{id}_X$ and $fg \simeq \mathrm{id}_Y$.

Homotopy Equivalence

Definition 30 (Homotopy Equivalence)

Two spaces X and Y are *homotopy equivalent*, written $X \simeq Y$, if and only if there exist maps

$$X \underset{g}{\overset{f}{\rightleftarrows}} Y$$

such that $gf \simeq \mathrm{id}_X$ and $fg \simeq \mathrm{id}_Y$.

Lemma 31

Homotopy equivalence is an equivalence relation on the collection of spaces.

Contractible Spaces

Definition 32 (Contractible)

A space X is *contractible* if and only if it is homotopy equivalent to the one-point space.

Contractible Spaces

Definition 32 (Contractible)

A space X is *contractible* if and only if it is homotopy equivalent to the one-point space.

Proposition 6

X is contractible iff $id_X \simeq c_x$ for some $x \in X$.

Examples: Convex subspaces of $\mathbb{R}^n$, D^n

Homotopy Retraction

Definition 33 (Homotopy Retract)

When A is a subspace of a space X and $i : A \to X$ is the inclusion map, $r : X \to A$ is called a *homotopy retract* if and only if $ri = \mathrm{id}_A$ and $ir \simeq id_X$.

In the above case, clearly $A \simeq X$.

Example: S^{n-1} and $\mathbb{R}^n - \{0\}$

Homotopy Relative to a Set

Definition 34 (Relative Homotopy)

Let X and Y be spaces and $A \subset X$ a subspace. Then $f, g : X \to Y$ are *homotopic relative to* A if and only if $f|_A = g|_A$ and there is a homotopy $H : f \simeq g$ such that $H(x,t) = f(x) = g(x)$ for all $x \in A, t \in I$.

Note that homotopy relative to a set is an equivalence relation and Lemma 29 holds in this case.

The Simplicial Approximation Theorem

Theorem 35 (Simplicial Approximation Theorem)

Let K and L be simplicial complexes, where K is finite, and $f : |K| \to |L|$ a continuous map. Then there exists a subdivision K' of K and simplicial map $g : K' \to L$ such that $|g| \simeq f$.

Hence, if we can triangulate a space, we can just think in terms of finite simplicial maps.

We need more machinery before we can prove this...

Simplicial Stars

Definition 36 (Star)

Let K be a simplicial complex and $x \in |K|$. The *star* of x in $|K|$, denoted $st_K(x)$, is defined as

$$st_K(x) = \bigcup \{\text{inside}(\sigma) : \sigma \text{ a simplex of K}, x \in \sigma\}$$

Simplicial Stars

Definition 36 (Star)

Let K be a simplicial complex and $x \in |K|$. The *star* of x in $|K|$, denoted $st_K(x)$, is defined as

$$st_K(x) = \bigcup \{\text{inside}(\sigma) : \sigma \text{ a simplex of K}, x \in \sigma\}$$

Lemma 37

For any $x \in |K|$, $st_K(x)$ is open in $|K|$.

Simplicial Stars

Proposition 7

Let K and L be simplicial complexes, and $f : |K| \to |L|$ be continuous. Suppose there exists a function $g : V(K) \to V(L)$ such that $f(st_K(v)) \subseteq st_L(g(v))$ for every $v \in V(K)$. Then g is a simplicial map and $|g| \simeq f$.

Simplicial Stars

Proposition 7

Let K and L be simplicial complexes, and $f : |K| \to |L|$ be continuous. Suppose there exists a function $g : V(K) \to V(L)$ such that $f(st_K(v)) \subseteq st_L(g(v))$ for every $v \in V(K)$. Then g is a simplicial map and $|g| \simeq f$.

Proposition 8

Let K, L, f and g be as in Proposition 7. Let A be a subcomplex of K and B a subcomplex of L, such that $f(|A|) \subseteq |B|$. Then $g(A) \subseteq B$ and the homotopy $H : |g| \simeq f$ sends $|A|$ to $|B|$ throughout, ie. $H(|A|, t) \subseteq |B|$ for all t.

Metrics on Simplices

We want a subdivision of K such that g exists as in Proposition 7. When is this possible? When the subdivision is 'sufficiently fine'...

Definition 38 (Standard Metric)

The *standard metric* d on a finite simplicial complex $|K|$ with vertices $\{v_0, v_1, \ldots, v_n\}$ is defined to be

$$d\left(\sum_i \lambda_i v_i, \sum_i \lambda_i' v_i\right) = \sum_i |\lambda_i - \lambda_i'|$$

This is clearly a metric on $|K|$.

Metrics on Simplices

Definition 39 (Coarseness)

Let K' be a subdivision of K. The *coarseness* of K' is

$$\sup\{d(x,y) : x, y \in st_K(v), v \text{ a vertex of } K'\}$$

Example: $(I \times I)_{(r)}$ has coarseness $4/r$ for $r \in \mathbb{N}$.

We want to show that g exists when the coarseness of K' is sufficiently small.

Aside - Covering Theorem

We will need the following from metric spaces:

Definition 40 (Diameter)

The *diameter* of a subset A of a metric space is defined as

$$\operatorname{diam}(A) = \sup\{d(x, y) : x, y \in A\}$$

Aside - Covering Theorem

We will need the following from metric spaces:

Definition 40 (Diameter)

The *diameter* of a subset A of a metric space is defined as

$$\mathrm{diam}(A) = \sup\{d(x, y) : x, y \in A\}$$

Theorem 41 (Lebesgue Covering Theorem)

Let X be a compact metric space and $\mathcal{C}$ an open covering of X. Then there exists a $\delta > 0$ such that every subset of X with diameter less than δ is entirely contained in some member of $\mathcal{C}$.

Back to the Main Theorem

An alternate phrasing we will first prove here:

Theorem 42

Let K, L be simplicial complexes, K finite, and $f : |K| \to |L|$ continuous. Then there exists a $\delta > 0$ such that for any subdivision K' of K with coarseness less than δ, there exists a simplicial map $g : K' \to L$ with $g \simeq f$.

A Minor Addendum

We append the following to Theorem 43, which we will need later:

Proposition 9

Let $A_1, \ldots, A_n$ be subcomplexes of K and $B_1, \ldots, B_n$ be subcomplexes of L such that $f(A_i) \subseteq B_i$ for all i. Then given the simplicial map g from Theorem 43, $|g|(A_i) \subseteq B_i$ and the homotopy $H : f \simeq |g|$ sends A_i to B_i throughout.

Finer Subdivisions

The Simplicial Approximation Theorem follows from Theorem 43 and the following:

Proposition 10

A finite simplicial complex K has subdivisions $K^{(r)}$ such that the coarseness of $K^{(r)}$ tends to 0 as $r \to \infty$.

The Fundamental Group

- A powerful tool to consider homotopic properties algebraically
- We will redefine this construct from the ground up
- Show a powerful conversion to a finite construction in terms of simplicial complexes
- Major result: fundamental groups of S^n are trivial for $n \geq 2$, and isomorphic to $\langle \mathbb{Z}, + \rangle$ for $n = 1$
- A surprising proof at the end...

Paths in a Space

Definition 43 (Path)

A *path* in a space X is a continuous map $f : I \to X$. A *loop based at a point* $b \in X$ is a path where $f(0) = f(1) = b$.

Alternatively, a loop is a continuous map $f : S^1 \to X$.

Paths in a Space

Definition 43 (Path)

A *path* in a space X is a continuous map $f : I \to X$. A *loop based at a point* $b \in X$ is a path where $f(0) = f(1) = b$.

Alternatively, a loop is a continuous map $f : S^1 \to X$.

Definition 44 (Composite Path)

Let X be a space and u, v paths in X such that $u(1) = v(0)$. The *composite path* $u.v$ is given by

$$u.v(t) = \begin{cases} u(2t) & \text{if } 0 \leq t \leq 1/2 \\ v(2t-1) & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

The Fundamental Group

We consider spaces with some basepoint $b \in X$, written $\langle X, b\rangle$. Continuous maps $f : \langle X, b\rangle \to \langle Y, c\rangle$ must have $f(b) = c$.

Definition 45 (Fundamental Group)

The *fundamental group of* $\langle X, b\rangle$, denoted $\pi_1(X, b)$, is the set of homotopy classes relative to ∂I of loops based at b, with the path composition operation.

The Fundamental Group

We consider spaces with some basepoint $b \in X$, written $\langle X, b \rangle$. Continuous maps $f : \langle X, b \rangle \to \langle Y, c \rangle$ must have $f(b) = c$.

Definition 45 (Fundamental Group)

The *fundamental group of* $\langle X, b \rangle$, denoted $\pi_1(X, b)$, is the set of homotopy classes relative to ∂I of loops based at b, with the path composition operation.

We still need to show this is a group!

Is $\pi_1(X, b)$ a Group?

Lemma 46 (Well-Definedness)

Suppose u and v are paths in X such that $u(1) = v(0)$, and u', v' are paths such that $u \simeq u', v \simeq v'$ relative to ∂I. Then $u.v \simeq u'.v'$ relative to ∂I.

Is $\pi_1(X, b)$ a Group?

Lemma 46 (Well-Definedness)

Suppose u and v are paths in X such that $u(1) = v(0)$, and u', v' are paths such that $u \simeq u', v \simeq v'$ relative to ∂I. Then $u.v \simeq u'.v'$ relative to ∂I.

Lemma 47 (Associativity)

Let u, v, w be paths in X such that $u(1) = v(0)$, $v(1) = w(0)$. Then $u.(v.w) \simeq (u.v).w$ relative to ∂I.

Is $\pi_1(X, b)$ a Group?

NB: $c_x : I \to X$ is the constant path at x.

Lemma 48 (Identity)

Let u be a path in X. Then $c_{u(0)}.u \simeq u \simeq u.c_{u(1)}$ relative to ∂I.

Is $\pi_1(X, b)$ a Group?

NB: $c_x : I \to X$ is the constant path at x.

Lemma 48 (Identity)

Let u be a path in X. Then $c_{u(0)}.u \simeq u \simeq u.c_{u(1)}$ relative to ∂I.

Lemma 49 (Inverses)

Let u be a path in X. Define u^{-1} to be the path such that $u^{-1}(t) = u(1-t)$ for all $t \in I$. Then $u.u^{-1} \simeq c_{u(0)}$ and $u^{-1}.u \simeq c_{u(1)}$ relative to ∂I.

Path-Components

Definition 50 (Path-Component)

A *path-component* of a space X is a maximal path-connected subset $A \subseteq X$.

The path-components of X partition the space.

Path-Components

Definition 50 (Path-Component)

A *path-component* of a space X is a maximal path-connected subset $A \subseteq X$.

The path-components of X partition the space.

Proposition 11

If $b, b' \in X$ are in the same path-component, then $\pi_1(X, b) \cong \pi_1(X, b')$.

If X is path-connected, we omit b and just write $\pi_1(X)$.

Induced Homomorphisms

Proposition 12

Let $\langle X, x\rangle$ and $\langle Y, y\rangle$ be spaces with basepoints. Then any continuous map $f : \langle X, x\rangle \to \langle Y, y\rangle$ induces a homomorphism $f_* : \pi_1(X, x) \to \pi_1(Y, y)$. Moreover:

1. $(id_X)_* = id_{\pi_1(X,x)}$
2. *if $g : \langle Y, y\rangle \to \langle Z, z\rangle$ is continuous, then $(gf)_* = g_* f_*$*
3. *if $f \simeq f'$ relative to $\{x\}$, then $f_* = f'_*$.*

Group Isomorphism

Theorem 51

Let X, Y be path-connected spaces with $X \simeq Y$. Then $\pi_1(X) \cong \pi_1(Y)$.

Group Isomorphism

Theorem 51

Let X, Y be path-connected spaces with $X \simeq Y$. Then $\pi_1(X) \cong \pi_1(Y)$.

Definition 52

A space is *simply-connected* if and only if it is path-connected and has trivial fundamental group.

A Simplicial Version

Definition 53 (Edge Path)

Let K be a simplicial complex. An *edge path* is a finite sequence $(a_0, \ldots, a_n)$ of vertices of K such that for each i, (a_{i-1}, a_i) spans a simplex of K. (Clearly (a_i, a_i) spans a 0-simplex.)

An *edge loop* is a path with $a_n = a_0$. We define edge composition by concatenation.

Elementary Contraction

Definition 54 (Elementary Contraction)

Let α be an edge path. An *elementary contraction* of α is an edge path obtained from α by performing one of the following moves:

1. Replace $(\dots, a_{i-1}, a_i, \dots)$ with $(\dots, a_i, \dots)$ if $a_{i-1} = a_i$;
2. Replace $(\dots, a_{i-1}, a_i, a_{i+1}, \dots)$ with $(\dots, a_i, \dots)$ if $a_{i-1} = a_{i+1}$;
3. Replace $(\dots, a_{i-1}, a_i, a_{i+1}, \dots)$ with $(\dots, a_{i-1}, a_{i+1}, \dots)$ if $\{a_{i-1}, a_i, a_{i+1}\}$ spans a 2-simplex of K.

An *elementary expansion* β of α is an edge path such that α is an elementary contraction of β.

Elementary Contraction

Definition 54 (Elementary Contraction)

Let α be an edge path. An *elementary contraction* of α is an edge path obtained from α by performing one of the following moves:

1. Replace $(\dots, a_{i-1}, a_i, \dots)$ with $(\dots, a_i, \dots)$ if $a_{i-1} = a_i$;
2. Replace $(\dots, a_{i-1}, a_i, a_{i+1}, \dots)$ with $(\dots, a_i, \dots)$ if $a_{i-1} = a_{i+1}$;
3. Replace $(\dots, a_{i-1}, a_i, a_{i+1}, \dots)$ with $(\dots, a_{i-1}, a_{i+1}, \dots)$ if $\{a_{i-1}, a_i, a_{i+1}\}$ spans a 2-simplex of K.

An *elementary expansion* β of α is an edge path such that α is an elementary contraction of β.

Note that rule 3 generalizes to any n-simplex contraction by contracting along the 2-faces.

Edge Loop Group

Definition 55 (Edge Equivalence)

Two edge paths α, β are said to be equivalent, written $\alpha \sim \beta$, if and only if β is the result of a finite series of elementary contractions and expansions applied to α.

Edge Loop Group

Definition 55 (Edge Equivalence)

Two edge paths α, β are said to be equivalent, written $\alpha \sim \beta$, if and only if β is the result of a finite series of elementary contractions and expansions applied to α.

Definition 56 (Edge Loop Group)

The *edge loop group* $E(K, b)$ for a given simplicial complex K and $b \in V(K)$ is the set of equivalence classes of loops over $\sim$ starting at b with the composition operation.

This is indeed a group, with identity (b) and inverses being the reversed path.

Triangulating Fundamental Groups

Theorem 57

For a simplicial complex K and vertex b, $E(K, b) \cong \pi_1(|K|, b)$.

Triangulating Fundamental Groups

Theorem 57

For a simplicial complex K and vertex b, $E(K,b) \cong \pi_1(|K|,b)$.

This clearly shows that fundamental groups can be made into finite, computable objects given a finite triangulation.

Triangulating Fundamental Groups

Theorem 57

For a simplicial complex K and vertex b, $E(K, b) \cong \pi_1(|K|, b)$.

This clearly shows that fundamental groups can be made into finite, computable objects given a finite triangulation.

Also, it shows $E(K, b)$ is independent of the choice of triangulation. So, it doesn't change with subdivisions.

Computing $\pi_1(S^n)$

Definition 58 (n-skeleton)

For a simplicial complex K and any non-negative integer n, the *n-skeleton of K*, denoted $\mathrm{skel}^n(K)$, is the subcomplex consisting of the simplicies with dimension $\leq n$.

Computing $\pi_1(S^n)$

Definition 58 (n-skeleton)

For a simplicial complex K and any non-negative integer n, the *n-skeleton of K*, denoted $\mathrm{skel}^n(K)$, is the subcomplex consisting of the simplicies with dimension $\leq n$.

Lemma 59

For any simplicial complex K and vertex b, $\pi_1(|K|, b) \cong \pi_1(|\mathrm{skel}^2(K)|, b)$.

Computing $\pi_1(S^n)$

Definition 58 (*n*-skeleton)

For a simplicial complex K and any non-negative integer n, the *n-skeleton of K*, denoted $\mathrm{skel}^n(K)$, is the subcomplex consisting of the simplicies with dimension $\leq n$.

Lemma 59

For any simplicial complex K and vertex b, $\pi_1(|K|, b) \cong \pi_1(|\mathrm{skel}^2(K)|, b)$.

Theorem 60

For $n \geq 2$, $\pi_1(S^n)$ is trivial.

Computing $\pi_1(S^n)$

Theorem 61

$\pi_1(S^1) \cong \langle \mathbb{Z}, + \rangle$.

The Fundamental Theorem of Algebra

You have seen FTA proven using Galois theory and with complex analysis. Here, we present a proof with algebraic topology.

Theorem 62 (Fundamental Theorem of Algebra)

For $f \in \mathbb{C}[X]$, $deg(f) > 0 \Rightarrow 0 \in f(\mathbb{C})$.

Free Groups

- We have shown existence of useful groups to topology; how do these groups look in general?
- Need a more formal concept of how to 'present' a group
- Idea: elements are *words over an alphabet* $S \cup S^{-1}$, where S is a generating set
- We will discover in doing this that the group-topology connection is two-way...

Words over S

We assume that the set S is such that $S \cap S^{-1} = \varnothing$, where $S^{-1} = \{s^{-1} \mid s \in S\}$. These are not inverses in any given group, just elements of S with an added $\cdot^{-1}$ superscript. We also specify that $(x^{-1})^{-1} = x$.

Definition 63 (Word)

For any set S, a *word* is a finite sequence $w = s_1 s_2 \ldots s_n$, where $s_n \in S \cup S^{-1}$.

Words over S

We assume that the set S is such that $S \cap S^{-1} = \varnothing$, where $S^{-1} = \{s^{-1} \mid s \in S\}$. These are not inverses in any given group, just elements of S with an added $\cdot^{-1}$ superscript. We also specify that $(x^{-1})^{-1} = x$.

Definition 63 (Word)

For any set S, a *word* is a finite sequence $w = s_1 s_2 \dots s_n$, where $s_n \in S \cup S^{-1}$.

Definition 64 (Concatenation)

For words $w_1 = s_1 \dots s_n, w_2 = r_1 \dots r_n$, the *concatenation* $w_1 w_2 = s_1 \dots s_n r_1 \dots r_n$.

Elementary Contractions

Definition 65 (Elementary Contraction/Expansion)

A word w' is an *elementary contraction* of a word w, written $w \searrow w'$, if $w = y_1 x x^{-1} y_2$ and $w' = y_1 y_2$ for words y_1, y_2 and $x, x^{-1} \in S \cup S^{-1}$.

A word w' is an *elementary expansion* of a word w, written $w \nearrow w'$, if $w' \searrow w$.

Elementary Contractions

Definition 65 (Elementary Contraction/Expansion)

A word w' is an *elementary contraction* of a word w, written $w \searrow w'$, if $w = y_1 x x^{-1} y_2$ and $w' = y_1 y_2$ for words y_1, y_2 and $x, x^{-1} \in S \cup S^{-1}$.

A word w' is an *elementary expansion* of a word w, written $w \nearrow w'$, if $w' \searrow w$.

Definition 66 (Word Equivalence)

Two words w, w' are *equivalent*, written $w \sim w'$, if and only if there exists a finite sequence of words $w = w_0, w_1, \ldots, w_n = w'$ such that $w_{i-1} \searrow w_i$ or $w_{i-1} \nearrow w_i$ for all i.

Free Group

Definition 67 (Free Group)

The *free group on the set* S, written $F(S)$, is the set of equivalence classes of words in the alphabet S with the concatenation operation.

This is clearly well-defined; $w \sim w', v \sim v' \Rightarrow wv \sim w'v'$. Checking the axioms is routine.

Free Group

Definition 67 (Free Group)

The *free group on the set S*, written $F(S)$, is the set of equivalence classes of words in the alphabet S with the concatenation operation.

This is clearly well-defined; $w \sim w', v \sim v' \Rightarrow wv \sim w'v'$. Checking the axioms is routine.

Definition 68 (Free Generating Set)

If for a group G there is an isomorphism $\theta : F(S) \to G$ for some set S, then $\theta(S)$ is known as a *free generating set*.

Reduced Representatives

We would like the 'minimal' version of a word if possible.

Definition 69 (Reduced)

A word is *reduced* if it permits no elementary contraction.

Reduced Representatives

We would like the 'minimal' version of a word if possible.

Definition 69 (Reduced)

A word is *reduced* if it permits no elementary contraction.

Lemma 70 (Sequential Independence)

Let w_1, w_2, w_3 be words such that $w_1 \searrow w_2 \nearrow w_3$. Then either $w_1 = w_3$ or there is a word w_2' such that $w_1 \nearrow w_2' \searrow w_3$.

Reduced Representatives

We would like the 'minimal' version of a word if possible.

Definition 69 (Reduced)

A word is *reduced* if it permits no elementary contraction.

Lemma 70 (Sequential Independence)

Let w_1, w_2, w_3 be words such that $w_1 \searrow w_2 \nearrow w_3$. Then either $w_1 = w_3$ or there is a word w_2' such that $w_1 \nearrow w_2' \searrow w_3$.

Theorem 71

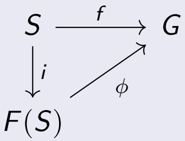
Any element of $F(S)$ is equivalent to a reduced word.

The Universal Property

Given a set S, there is a *canonical inclusion* $i : S \to F(S)$, namely the identity.

Theorem 72 (Universal Property)

Given any set S, any group G and function $f : S \to G$, there is a unique homomorphism $\phi : F(S) \to G$ such that the following diagram commutes:



Fundamental Groups of Graphs

An immediate interesting application of free groups to topology: graphs!
Any graph can be seen as a topology by considering the equivalent 1-dimensional cell complex.

Fundamental Groups of Graphs

An immediate interesting application of free groups to topology: graphs!
Any graph can be seen as a topology by considering the equivalent 1-dimensional cell complex.

Theorem 73

The fundamental group of a countable connected graph is free.

Fundamental Groups of Graphs

An immediate interesting application of free groups to topology: graphs!
Any graph can be seen as a topology by considering the equivalent 1-dimensional cell complex.

Theorem 73

The fundamental group of a countable connected graph is free.

We will spend the rest of today proving this.

Subgraphs and Edge Paths

Definition 74 (Subgraph)

Let Γ be a graph with vertex set V, edge set E, and endpoint function δ. A *subgraph* of Γ is a graph with vertex set $V' \subseteq V$, edge set $E' \subseteq E$ and $\delta' = \delta\,|_{E'}$ such that $\bigcup \delta'(E') \subseteq V'$.

Subgraphs and Edge Paths

Definition 74 (Subgraph)

Let Γ be a graph with vertex set V, edge set E, and endpoint function δ. A *subgraph* of Γ is a graph with vertex set $V' \subseteq V$, edge set $E' \subseteq E$ and $\delta' = \delta\,|_{E'}$ such that $\bigcup \delta'(E') \subseteq V'$.

Clearly if Γ is oriented, the orientation can similarly be inherited.

Subgraphs and Edge Paths

Definition 74 (Subgraph)

Let Γ be a graph with vertex set V, edge set E, and endpoint function δ. A *subgraph* of Γ is a graph with vertex set $V' \subseteq V$, edge set $E' \subseteq E$ and $\delta' = \delta\,|_{E'}$ such that $\bigcup \delta'(E') \subseteq V'$.

Clearly if Γ is oriented, the orientation can similarly be inherited.

Definition 75 (Edge Path)

An *edge path* in a graph Γ is a path concatenation $u_0.u_1.\dots.u_n$, where each u_i is either a path running along a single edge at unit speed or a constant path based at a vertex. An *edge loop* is an edge path where $u(0) = u(1)$. An edge path (loop) $u : I \to \Gamma$ is *embedded* if u is injective (injective on I^o.)

Trees

Definition 76 (Tree)

A *tree* is a connected graph with no embedded edge loops.

Trees

Definition 76 (Tree)

A *tree* is a connected graph with no embedded edge loops.

Lemma 77

In a tree, there is a unique embedded edge path between distinct vertices.

Maximal Trees

Definition 78 (Maximal Tree)

A *maximal tree* of a connected graph Γ is a subgraph T that is a tree, but adding any edge in $E_\Gamma \setminus E_T$ creates an embedded edge loop.

Maximal Trees

Definition 78 (Maximal Tree)

A *maximal tree* of a connected graph Γ is a subgraph T that is a tree, but adding any edge in $E_\Gamma \setminus E_T$ creates an embedded edge loop.

Lemma 79

Let Γ be a connected graph and T be a subgraph that is a tree. Then the following are equivalent:

1. $V_T = V_\Gamma$;
2. T *is maximal.*

Maximal Trees

Lemma 80

Any connected countable graph Γ contains a maximal tree.

Maximal Trees

Lemma 80

Any connected countable graph Γ contains a maximal tree.

(Aside - This is only true for uncountable graphs if we accept the Axiom of Choice. However, we won't ever need the uncountable case.)

Maximal Trees

Lemma 80

Any connected countable graph Γ contains a maximal tree.

(Aside - This is only true for uncountable graphs if we accept the Axiom of Choice. However, we won't ever need the uncountable case.)

With this, we can finally prove Theorem 73, namely that every countable connected graph has free fundamental group.

Maximal Trees

Lemma 80

Any connected countable graph Γ contains a maximal tree.

(Aside - This is only true for uncountable graphs if we accept the Axiom of Choice. However, we won't ever need the uncountable case.)

With this, we can finally prove Theorem 73, namely that every countable connected graph has free fundamental group.

Examples: n-bouquet, Cayley graph of $\mathbb{Z}^2$ with generating set $\{(0,1),(1,0)\}$

Group Presentations

- It's time to develop a way to 'write out' any group
- Groups can be seen as a free group where some words are identified (eg. D_{2n}); makes many infinite groups possible to reason about finitely
- When are two presentations equal?
- What are the presentations of fundamental groups?
- End this lecture series on a high note - a deep connection between group presentations and topological spaces

Generating Normal Subgroups

Definition 81 (Normal Subgroup Generated by B)

Let $B \subseteq G$, where G is a group. The *normal subgroup generated by* B is the intersection of all normal subgroups containing B, denoted $\langle B \rangle$.

Generating Normal Subgroups

Definition 81 (Normal Subgroup Generated by B)

Let $B \subseteq G$, where G is a group. The *normal subgroup generated by* B is the intersection of all normal subgroups containing B, denoted $\langle B \rangle$.

Proposition 13

The subgroup $\langle B \rangle$ *consists of all expressions of the form*

$$\prod_{i=1}^{n} g_i b_i^{\epsilon_i} g_i^{-1}$$

for $n \in \mathbb{Z}_0$, $g_i \in G$, $b_i \in B$ *and* $\epsilon_i = \pm 1$ *for all* i.

Group Presentations

Definition 82 (Presentation)

Let X be a set, and $R \subseteq F(X)$. The *group with presentation* $\langle X \mid R \rangle$ is defined as $F(X)/\langle R \rangle$.

Group Presentations

Definition 82 (Presentation)

Let X be a set, and $R \subseteq F(X)$. The *group with presentation* $\langle X \mid R \rangle$ is defined as $F(X)/\langle R \rangle$.

Example: Dihedral group $D_{2n} = \langle \sigma, \tau \mid \sigma^n, \tau^2, \tau\sigma\tau\sigma \rangle$

Group Presentations

Definition 82 (Presentation)

Let X be a set, and $R \subseteq F(X)$. The *group with presentation* $\langle X \mid R \rangle$ is defined as $F(X)/\langle R \rangle$.

Example: Dihedral group $D_{2n} = \langle \sigma, \tau \mid \sigma^n, \tau^2, \tau\sigma\tau\sigma \rangle$

Natural question: when are two words w, w' equivalent in $\langle X \mid R \rangle$? We call this the *word problem*.

Group Presentations

Definition 82 (Presentation)

Let X be a set, and $R \subseteq F(X)$. The *group with presentation* $\langle X \mid R \rangle$ is defined as $F(X)/\langle R \rangle$.

Example: Dihedral group $D_{2n} = \langle \sigma, \tau \mid \sigma^n, \tau^2, \tau\sigma\tau\sigma \rangle$

Natural question: when are two words w, w' equivalent in $\langle X \mid R \rangle$? We call this the *word problem*.

Proposition 14

Two words $w, w' \in F(X)$ are equal in $\langle X \mid R \rangle$ if and only if they differ by a finite number of the following operations:

1. *Elementary contractions or expansions;*
2. *Inserting an element of $\langle R \rangle$ into one of the words.*

Group Presentations

Definition 83 (Finite Presentation)

A presentation $\langle X \mid R\rangle$ is *finite* if and only if X and R are finite. Likewise, a group is *finitely presented* if it has a finite presentation.

Group Presentations

Definition 83 (Finite Presentation)

A presentation $\langle X \mid R \rangle$ is *finite* if and only if X and R are finite. Likewise, a group is *finitely presented* if it has a finite presentation.

Aside: There is a rewriting system such that any two finite presentations present the same group iff they can be rewritten to each other in this system. Called *Tietze transformations*.

Group Presentations

Definition 83 (Finite Presentation)

A presentation $\langle X \mid R \rangle$ is *finite* if and only if X and R are finite. Likewise, a group is *finitely presented* if it has a finite presentation.

Aside: There is a rewriting system such that any two finite presentations present the same group iff they can be rewritten to each other in this system. Called *Tietze transformations*.

Proposition 15

Let $\langle X \mid R \rangle$, H be groups. Let $f : X \to H$ induce a homomorphism $\phi : F(X) \to H$. This descends to a homomorphism $\langle X \mid R \rangle \to H$ if and only if $\phi(R) = \{1_H\}$, ie. $R \subseteq \ker(\phi)$.

Push-outs

Definition 84 (Push-out)

Let G_0, G_1, G_2 be groups and $\phi_1 : G_0 \to G_1$, $\phi_2 : G_0 \to G_2$ be homomorphisms. Let $\langle X_1 \mid R_1 \rangle$ and $\langle X_2 \mid R_2 \rangle$ be presentations of G_1, G_2 respectively where $X_1 \cap X_2 = \varnothing$.
The *push-out* $G_1 *_{G_0} G_2$ of

$$G_1 \overset{\phi_1}{\leftarrow} G_0 \overset{\phi_2}{\rightarrow} G_2$$

is the group

$$\langle X_1 \cup X_2 \mid R_1 \cup R_2 \cup \{\phi_1(g) = \phi_2(g) \mid g \in G_0\} \rangle$$

Push-outs are independent of the G_1, G_2 presentations (proof omitted.)

Push-outs

Proposition 16 (Universal Property)

Given a pushout $G_1 *_{G_0} G_2$ of

$$G_1 \overset{\phi_1}{\leftarrow} G_0 \overset{\phi_2}{\rightarrow} G_2$$

and a group H with morphisms $\beta_i : G_i \to H$ such that the diagram

$$\begin{array}{ccc} G_0 & \xrightarrow{\phi_1} & G_2 \\ {\scriptstyle \phi_2}\downarrow & & \downarrow{\scriptstyle \beta_2} \\ G_1 & \xrightarrow{\beta_1} & H \end{array}$$

commutes, then there exists a unique homomorphism $\phi : G_1 *_{G_0} G_2 \to H$ such that the above diagram together with $G_1 \to G_1 *_{G_0} G_2 \leftarrow G_2$ commutes.

Push-outs

Definition 85 (Free Product)

When G_0 in our definition of a push-out is trivial, the push-out is called the *free product* of G_1 and G_2.

Definition 86 (Amalgamated Free Product)

When $\phi_1 : G_0 \to G_1$ and $\phi_2 : G_0 \to G_2$ are injective, we say the push-out is the *amalgamated free product* of G_1 and G_2 along G_0.

Push-outs of Fundamental Groups

Theorem 87 (Seifert - van Kampen Theorem)

Let K be a space which is a union of path-connected open sets K_1, K_2, where $K_1 \cap K_2$ is path-connected. Then for $b \in K_1 \cap K_2$ and $i_x : K_1 \cap K_2 \to K_x$ the inclusion maps, we have

$$\pi_1(K, b) \cong \pi_1(K_1, b) *_{\pi_1(K_1 \cap K_2, b)} \pi_1(K_2, b)$$

Moreover, the homomorphisms $\pi_1(K_i, b) \to \pi_1(K, b)$ are the maps induced by inclusion.

Push-outs of Fundamental Groups

Theorem 87 (Seifert - van Kampen Theorem)

Let K be a space which is a union of path-connected open sets K_1, K_2, where $K_1 \cap K_2$ is path-connected. Then for $b \in K_1 \cap K_2$ and $i_x : K_1 \cap K_2 \to K_x$ the inclusion maps, we have

$$\pi_1(K, b) \cong \pi_1(K_1, b) *_{\pi_1(K_1 \cap K_2, b)} \pi_1(K_2, b)$$

Moreover, the homomorphisms $\pi_1(K_i, b) \to \pi_1(K, b)$ are the maps induced by inclusion.

This gives us a way to build presentations of $\pi_1(K, b)$ from smaller parts.

Topological Application

Recall that conjugacy classes in $\pi_1(K, b)$ correspond to homotopy classes of baseless loops in K.

Topological Application

Recall that conjugacy classes in $\pi_1(K, b)$ correspond to homotopy classes of baseless loops in K.

Theorem 88

Let K be a connected cell complex, and let $l_i : S^1 \to K^1$ be the attaching maps of its 2-cells, where $1 \leq i \leq n$. Let b be a basepoint in K^0. Let $[l_i]$ be the conjugacy class of the loop l_i in $\pi_1(K^1, b)$. Then

$$\pi_1(K, b) \cong \pi_1(K^1, b)/\langle [l_1] \cup \cdots \cup [l_n] \rangle.$$

Topological Application

Recall that conjugacy classes in $\pi_1(K, b)$ correspond to homotopy classes of baseless loops in K.

Theorem 88

Let K be a connected cell complex, and let $l_i : S^1 \to K^1$ be the attaching maps of its 2-cells, where $1 \leq i \leq n$. Let b be a basepoint in K^0. Let $[l_i]$ be the conjugacy class of the loop l_i in $\pi_1(K^1, b)$. Then

$$\pi_1(K, b) \cong \pi_1(K^1, b)/\langle [l_1] \cup \cdots \cup [l_n] \rangle.$$

Example: $\mathbb{T}^2$.

From Presentations to Spaces

The major result of this course:

Theorem 89

The following are equivalent for a group G:

1. *G is finitely presented;*
2. *G is the fundamental group of a finite connected cell complex;*
3. *G is the fundamental group of a finite connected simplicial complex.*