

# Homotopy Bicategories of Complete 2-fold Segal Spaces

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## Abstract

In this paper, we address the construction of homotopy bicategories of  $(\infty, 2)$ -categories, which we take as being modeled by complete 2-fold Segal spaces. Our main result is the concrete construction of a functor  $h_2$  from the category of Reedy fibrant complete 2-fold Segal spaces (each decorated with chosen sections of the Segal maps) to the category of unbiased bicategories and pseudofunctors between them. Though our construction depends on the choice of sections, we show that for a given complete 2-fold Segal space, all possible choices yield the same unbiased bicategory up to an equivalence that acts as the identity on objects, morphisms and 2-morphisms. We illustrate our results with the example of the fundamental bigroupoid of a topological space.

## 1 Introduction

The development of higher category theory has proven to follow a complete reversal of the usual timeline for mathematical research. One should usually expect to define the constructions of interest and later prove theorems they satisfy. Higher category theorists have instead found themselves on a different trajectory, developing definitions with the intent of satisfying certain theorems. In the face of such a conundrum, one may find themselves questioning what is theorem or definition. For instance, a desirable property of higher groupoids is the *homotopy hypothesis*, posed by Grothendieck in [20], which declares that  $n$ -groupoids should model homotopy  $n$ -types. Some models of higher category prove this as a theorem, like Bénabou's bicategories [5] or the groupoidal weakly globular  $n$ -fold categories of Paoli [38], while many instead take it as an axiom in their definitions, such as quasicategories [28], complicial sets [45], Segal  $n$ -categories [25], complete  $n$ -fold Segal spaces [3],  $\Theta_n$ -spaces [8] and  $n$ -quasi-categories [1].

Due in part to these variable foundations, it has become somewhat challenging to establish translations between all the diverse definitions of higher category. In exchange however, each correspondence constructed sheds new light on both models being considered, as one must devise means to extract definition from theorem or theorem from definition. One example of such a correspondence is that of a *homotopy category*. Take

an  $\infty$ -groupoid  $X$ . Should we believe the homotopy hypothesis, where the objects of  $X$  may be seen as points, 1-cells as paths and so on in some topological space that we may write as  $|X|$ , one may consider the set of path components  $\pi_0(X) \simeq \pi_0(|X|)$ , which is just the set of equivalence classes of objects in  $X$ . A somewhat frivolous name for this could be the ‘homotopy set’ of  $X$ . In a similar manner, one may consider an  $(\infty, 1)$ -category  $Y$ , where the  $\infty$ -category of morphisms  $\mathbf{Hom}_Y(x, y)$  for any objects  $x, y \in Y$  is an  $\infty$ -groupoid. We may use  $\pi_0$  to obtain a category from  $Y$ , called the homotopy category, whose objects are the same and whose hom-sets take the form  $\pi_0(\mathbf{Hom}_Y(x, y))$ .

In general, there should be a means to construct from any  $(\infty, n)$ -category  $X$  its *homotopy  $n$ -category*  $h_n(X)$ , for  $n \geq 0$ , by taking the  $k$ -cells of  $h_n(X)$  to be those of  $X$  for  $k < n$  and the  $n$ -cells in  $h_n(X)$  to be equivalence classes of  $n$ -cells in  $X$ . Unfortunately, such a construction remains elusive<sup>1</sup>, owing in no small part to the monstrously complex coherence conditions for a weak  $n$ -category. Some cases have already been established. ‘Homotopy sets,’ more commonly just referred to as sets of path components, are easily constructed from Kan complexes. Homotopy categories of quasicategories are classical [33, pg. 24], while Rezk has established homotopy categories for complete Segal spaces [42]. Campbell has developed homotopy bicategories of 2-quasi-categories [12], building on work of Lack and Paoli translating between Tamsamani 2-categories and bicategories [30]. Campbell also notes in [12] many ‘nerve’ functors which should be right adjoint to suitable notions of homotopy bicategory, such as the functor from strict 2-categories to quasicategory-enriched categories defined by enriching along the nerve functor of quasicategories, the nerve of a bicategory as a 2-dimensional Postnikov complex due to Duskin [14] and Gurski [21], the nerve of strict 2-category as a 2-precomplicial set due to Ozornova and Rovelli [37] and the composition of Lack and Paoli’s nerve for bicategories [30] with change of base by the quasicategory nerve functor to obtain Reedy fibrant quasicategory-enriched Segal categories. Johnson-Freyd, Calaque and Scheimbauer have suggested a notion of homotopy bicategory for projective fibrant complete 2-fold Segal spaces in [27] and [11].

In this paper, we continue the mission of establishing constructions of homotopy  $n$ -category in the case  $n = 2$ . In particular, we take *Reedy fibrant complete 2-fold Segal spaces* as defined in [27] as our model of an  $(\infty, 2)$ -category. These are bisimplicial spaces  $X : (\Delta^{op})^2 \rightarrow \mathbf{sSet}$  satisfying some conditions. We should interpret  $X_0$  as the ‘underlying  $(\infty, 1)$ -category of  $X$ ’, so the objects of  $X$  are the set  $(X_{0,0})_0$ ,  $X_1$  as the ‘ $(\infty, 1)$ -category of 1-morphisms in  $X$ ’, and  $X_k$  as the ‘ $(\infty, 1)$ -category of chains of 1-morphisms of length  $k$  in  $X$ , together with all compositions.’ One of the conditions to be a complete 2-fold Segal space is that the simplicial maps  $X_n \rightarrow X_1 \times_{X_0} \cdots \times_{X_0} X_1$ , often called the *Segal maps*, are trivial fibrations in a suitable model structure. This map should be interpreted as sending an element of  $X_k$  to the underlying chain of 1-morphisms in  $X$ . That this is a trivial fibration implies the fiber over any chain is contractible in a suitable sense. This implies that any chain has a composite which is unique up to higher homotopies.

Our contribution is to define a functor

$$h_2 : \mathbf{CSSP}_2^{comp} \rightarrow \mathbf{UBicat}$$

from the category  $\mathbf{CSSP}_2^{comp}$  of these Reedy fibrant complete 2-fold Segal spaces, decorated with chosen sections of the Segal maps identifying choices of compositions, to the

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<sup>1</sup>There is a notion of homotopy  $n$ -category for general  $n$  for quasicategories, though this is just a matter of truncating higher cells without translating to a meaningfully different definition of  $n$ -category. We therefore do not consider it here.

category **UBicat** *unbiased bicategories* and *unbiased pseudofunctors*<sup>2</sup>, as defined in [31]. Our construction has the advantage of being entirely explicit, demonstrating directly how to obtain composition maps, associators and unitors from lifting problems. This approach yields insight into how the assumptions of the homotopy hypothesis and Segal conditions used to define complete 2-fold Segal spaces generate the coherence structures in a notion of bicategory that does not make such assumptions. Of course, there will in general be many solutions to these lifting problems, yet functoriality of our construction ensures that all choices yield the same unbiased bicategory up to a pseudofunctor which is the identity on objects, morphisms and 2-morphisms.

It is our hope that the results of this paper should shed light on how ‘algebraic’ definitions of higher category, such as unbiased bicategories, may be induced by ‘non-algebraic’ ones, like complete 2-fold Segal spaces, by solving particular natural lifting problems. Future work will be to extend the results here to homotopy  $n$ -categories for general  $n$ , using a definition of  $n$ -category such as the Trimble  $n$ -categories discussed in [13], as well as to consider extra structure on domain and codomain, such as symmetric monoidal structures. The latter will be of particular application to fully extended topological quantum field theories, allowing the definitions of  $(\infty, 2)$ -category of manifolds and cobordisms defined by Lurie [34] and Calaque and Scheimbauer [11] to be translated to the symmetric monoidal bicategories given by Schommer-Pries [44], Pstrągowski [39] and others. We also take interest in producing a right adjoint to our construction and demonstrating a Quillen adjunction, as well as in accounting for pseudonatural transformations to obtain a 2-adjunction.

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<sup>2</sup>Leinster names these *unbiased weak functors* in [31].

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## 2 Complete 2-fold Segal Spaces

In producing a comparison between two models of higher category, our first port of call is to establish the fine-print of the models we have chosen. Our starting point is that of  $(\infty, 2)$ -categories, which for us, are *complete 2-fold Segal spaces*.

Our choice of complete 2-fold Segal spaces as a model for  $(\infty, 2)$ -categories comes largely from their connections to fully extended topological quantum field theories [27] [11] [34], along with their capacity to be inductively extended to a model of  $(\infty, n)$ -category [3] [9] [11] [32] [4]; extending the constructions in this paper to general  $n$  will be future work. The form of homotopy bicategory presented here is also a natural extension of the construction discussed in [11] and [27], though applied this time to Reedy fibrant complete 2-fold Segal spaces so that certain lifting problems are made available.

### 2.1 Simplicial Sets

Before we can proceed much further, we should take the time to establish some of the important infrastructure for our definitions. Our formalism is inspired by the discussion in [42, pg. 5-7], though primarily applies results from [24]. It should be noted thus that all our model categories are assumed to have functorial factorizations, as in [24, Def. 7.1.3].

**Definition 2.1.** *The simplicial category  $\Delta$  is the full subcategory of  $\mathbf{Cat}$  whose objects are the posets  $[n] := \{0 < \cdots < n\}$  for all  $n \geq 0$ .*

We choose to write  $\langle f_0, \cdots, f_m \rangle : [m] \rightarrow [n]$  for the map in  $\Delta$  such that  $i \mapsto f_i$  for all  $0 \leq i \leq m$ . There are special maps

$$d_i^m := \langle 0, \cdots, i-1, i+1, \cdots, n \rangle : [n] \rightarrow [n+1]$$

which we dub the *coface maps*, and

$$s_i^n := \langle 0, \cdots, i, i, \cdots, n \rangle : [n] \rightarrow [n-1]$$

which we call the *codegeneracy maps*. We will often omit the superscript when it is evident.

**Definition 2.2.** *The category of simplicial sets  $\mathbf{sSet} := \mathbf{Set}^{\Delta^{op}}$  is the category of presheaves on  $\Delta$ .*

In general, for any *simplicial object*  $X$  in a category  $\mathcal{C}$ , meaning a functor  $X : \Delta^{op} \rightarrow \mathcal{C}$ , we write  $X_i := X([i])$  and  $X_f : X_i \rightarrow X_j$  for  $X(f)$ , given a map  $f : [j] \rightarrow [i]$  in  $\Delta$ . For instance, there are maps  $X_{d_0}, X_{d_1}, X_{d_2} : X_2 \rightarrow X_1$  and  $X_{(0,4,7)} : X_9 \rightarrow X_2$ . For a simplicial set  $X$ , we call the maps  $X_{d_i^n}$  the *face maps* and  $X_{s_i^n}$  the *degeneracy maps*.

Write  $\Delta[n]$  for the representable simplicial set from  $[n]$  and  $\partial\Delta[n]$  for the maximal subobject of  $\Delta[n]$  not containing the identity  $1_{[n]} : [n] \rightarrow [n]$ . Here, a ‘subobject’ simply means a levelwise subset.

**Proposition 2.1** ([24, Example 9.1.13]).  *$\mathbf{sSet}$  is a simplicial model category with the simplicial mapping spaces, for any  $X, Y \in \mathbf{sSet}$ , given by*

$$(Y^X)_n := \mathbf{Hom}_{\mathbf{sSet}}(X \times \Delta[n], Y)$$

with the evident simplicial maps.

This structure in fact makes  $\mathbf{sSet}$  Cartesian closed - indeed, it is a case of the Cartesian closed structure on the presheaf category for any small domain category described in [35, pg. 46].

Note that there is a fully faithful embedding  $\iota_{\mathbf{s}} : \mathbf{Set} \hookrightarrow \mathbf{sSet}$ , sending a set  $X$  to a levelwise constant simplicial set. We call such simplicial sets *discrete*.

There is an important comparison between simplicial sets and topological spaces. Henceforth, let  $\mathbf{Top}$  be a convenient category of topological spaces, such as  $k$ -spaces.

**Definition 2.3.** *There is a functor  $\bullet_t : \Delta \rightarrow \mathbf{Top}$  sending  $[n]$  to the space*

$$\Delta_t[n] := \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid \sum_i x_i = 1, x_i \geq 0\}.$$

and  $f : [n] \rightarrow [m]$  to the unique affine map  $\Delta_t[n] \rightarrow \Delta_t[m]$  sending each coordinate vector  $e_i \in \mathbb{R}^{n+1}$  to  $e_{f_i} \in \mathbb{R}^{m+1}$ .

**Definition 2.4.** *Define the functor  $\mathbf{Sing} : \mathbf{Top} \rightarrow \mathbf{sSet}$  to send  $X \in \mathbf{Top}$  to the simplicial set  $\mathbf{Sing}(X)$ , defined levelwise as*

$$\mathbf{Sing}(X)_n := \mathbf{Hom}_{\mathbf{Top}}(\Delta_t[n], X).$$

For a map  $f : X \rightarrow Y$  in  $\mathbf{Top}$ ,  $\mathbf{Sing}(f)$  is given by levelwise postcomposition, sending an  $n$ -simplex  $\Delta_t[n] \rightarrow X$  to its composition with  $f : X \rightarrow Y$ .

By [35, pg. 41, Theorem 2], this functor has a left adjoint  $|\bullet|$ , referred to as the *geometric realisation functor*, which restricts on  $\Delta$  to  $\bullet_t$ . We take little interest in it here. What is of interest however is that these functors form a Quillen equivalence between the Quillen model structure on  $\mathbf{sSet}$  and the Quillen model structure on  $\mathbf{Top}$  [19, Thm. 10.10] [19, Thm. 11.4].

For a simplicial set  $X$ , we will call the elements of  $X_n$  the  *$n$ -simplices* of  $X$ .

**Definition 2.5** ([23, pg. 60]). *Let  $k \geq 0$  and  $X \in \mathbf{sSet}$ . Then the  $k$ -skeleton of  $X$ , written  $\mathbf{sk}_k X$ , is the minimal subobject of  $X$  such that  $(\mathbf{sk}_k)_n = X_n$  for all  $0 \leq n \leq k$ .*

We will need more substantial data structures than mere simplicial sets:

**Definition 2.6.** *The category of  $n$ -uple simplicial spaces, for any  $n \geq 0$ , is defined to be  $\mathbf{sSpace}_n := \mathbf{sSet}^{(\Delta^{op})^n}$ .*

*The category of simplicial spaces is the category of 1-uple simplicial spaces, which we write as  $\mathbf{sSpace} := \mathbf{sSpace}_1$ .*

*The category of bisimplicial spaces is the category of 2-uple simplicial spaces, which we write as  $\mathbf{ssSpace} := \mathbf{sSpace}_2$ .*

Write  $\Delta[i_0, \dots, i_n]$  for the  $n$ -uple simplicial space represented by  $([i_0], \dots, [i_n]) \in \Delta^{n+1}$ , noting  $\mathbf{sSet}^{(\Delta^{op})^n} \cong \mathbf{Set}^{(\Delta^{op})^{n+1}}$ . Our convention for ordering of indices follows from the evident isomorphism

$$\mathbf{sSet}^{(\Delta^{op})^n} \cong (\dots (\mathbf{sSet}^{\Delta^{op}}) \dots)^{\Delta^{op}}$$

which does not permute the copies of  $\Delta^{op}$ . With this in mind we also write, for  $X \in \mathbf{sSpace}_n$ , that

$$X_{i_1, \dots, i_n} := ((X_{i_1}) \dots)_{i_n}.$$

Hence, by the Yoneda lemma,

$$\mathbf{Hom}_{\mathbf{sSpace}_n}(\Delta[i_0, \dots, i_n], X) \cong (X_{i_n, i_{n-1}, \dots, i_1})_{i_0}.$$

**Proposition 2.2.** *The categories  $\mathbf{sSpace}_n$  are Cartesian closed. This moreover makes them symmetric monoidal.*

*Proof.* These categories are all categories of presheaves. □

**Definition 2.7.** *Let  $n \geq 1$ . Define the map*

$$\rho_n : \mathbf{sSpace}_n \rightarrow \mathbf{sSpace}_{n-1}$$

*so that  $\rho_n(X) = X_0$ . More precisely,  $\rho_n$  is defined by the precomposition map on presheaf categories induced by the functor  $\Delta^n \rightarrow \Delta^{n+1}$  sending*

$$([i_0], \dots, [i_{n-1}]) \mapsto ([i_0], \dots, [i_{n-1}], [0]).$$

**Notation 2.1.** *Write*

$$\rho_n^{n-k} := \rho_{k+1} \circ \dots \circ \rho_{n-1} \circ \rho_n.$$

*Hence,  $\rho_n = \rho_n^{n-1}$ .*

**Definition 2.8.** *Let  $0 \leq k < n$ . Define the map*

$$F_n^k : \mathbf{sSpace}_k \rightarrow \mathbf{sSpace}_n$$

*such that, for any  $K \in \mathbf{sSpace}_k$ ,*

$$(F_n^k(K)_{i_1, \dots, i_n})_j = (K_{i_1, \dots, i_k})_{i_{k+1}}.$$

*More formally, this is given by the precomposition map on presheaf categories induced by the functor  $\Delta^{n+1} \rightarrow \Delta^{k+1}$  sending*

$$([i_0], \dots, [i_n]) \mapsto ([i_{n-k}], \dots, [i_n]).$$

Note that  $F_n^0(K)$  is levelwise a discrete  $(n - 1)$ -uple simplicial space.

We will also need to understand boundaries of our representable  $n$ -uple simplicial spaces, once we discuss model structures:

**Notation 2.2.** Write  $\partial\Delta[i_n, \dots, i_0]$  for the maximal sub-presheaf of  $\Delta[i_n, \dots, i_0]$  missing the element  $(1_{[i_n]}, \dots, 1_{[i_0]})$ .

**Notation 2.3.** Write

$$\begin{aligned} F_n^k(i_k, \dots, i_0) &= F_n^k(\Delta[i_k, \dots, i_0]) \\ \partial F_n^k(i_k, \dots, i_0) &= F_n^k(\partial\Delta[i_k, \dots, i_0]). \end{aligned}$$

**Lemma 2.1.** There is an isomorphism

$$F_n^k(i_k, \dots, i_0) \cong \Delta[0, \dots, 0, i_k, \dots, i_0].$$

**Proposition 2.3.** For all  $n \geq 1$  and  $i_0, \dots, i_n \geq 0$ , there is an isomorphism

$$\partial\Delta[i_n, \dots, i_0] \cong (\Delta[i_n, \dots, i_1, 0] \times \partial F_n^0(i_0)) \sqcup_{\partial\Delta[i_n, \dots, i_1, 0] \times \partial F_n^0(i_0)} (\partial\Delta[i_n, \dots, i_1, 0] \times F_n^0(i_0)).$$

*Proof.* The pushout contains all maps in  $\Delta^n$  of the form  $([j_n], \dots, [j_0]) \rightarrow ([i_n], \dots, [i_0])$  such that either  $[j_0] \rightarrow [i_0]$  is not a surjection or such that the maps  $[j_h] \rightarrow [i_h]$  are not all surjections for  $1 \leq h \leq n$ . Thus, all maps which are not surjections on every coordinate are contained, which is an alternative characterization of  $\partial\Delta[i_n, \dots, i_0]$ .  $\square$

**Proposition 2.4.** For any  $n$ , the functor  $\rho_n^{n-k}$  is symmetric monoidal with respect to the Cartesian closed structures.

*Proof.*  $\rho_n$  is an inverse image functor between categories of presheaves.  $\square$

**Proposition 2.5.** By the functor  $\rho_n^{n-k}$  for any  $0 \leq k \leq n$ , the category  $\mathbf{sSpace}_n$  is enriched in  $\mathbf{sSpace}_k$ .

*Proof.* Apply the change of base results for enriched categories in [43, Lemma 3.4.3] and the enrichment of  $\mathbf{sSpace}_k$  in itself by the Cartesian structure.  $\square$

**Notation 2.4.** For any  $0 \leq k \leq n$  and some  $X, Y \in \mathbf{sSpace}_n$ , write  $\mathbf{Map}_n^k(X, Y)$  for the induced  $k$ -uple mapping space between  $X$  and  $Y$ . More concretely, this is

$$\mathbf{Map}_n^k(X, Y) = (Y^X)_{\underbrace{0, \dots, 0}_{n-k}}.$$

If  $n = k$ , this is just the inner hom.

There is an interesting complementary interaction here between  $F_n^k$  and  $\rho_n^{n-k-1}$ :

**Proposition 2.6.** For any  $0 \leq k < n$  and  $i_0, \dots, i_k \geq 0$ ,

$$\mathbf{Map}_n^{n-k-1}(F_n^k(i_k, \dots, i_0), X) \cong X_{i_0, \dots, i_k}.$$

*Proof.* We can compare the two sides levelwise. For  $j_0, \dots, j_{n-k-1} \geq 0$ :

$$\begin{aligned}
& (\mathbf{Map}_n^{n-k-1}(F_n^k(\Delta[i_k, \dots, i_0]), X)_{j_0, \dots, j_{n-k-2}})_{j_{n-k-1}} = ((X^{F_n^k(\Delta[i_k, \dots, i_0])})_{0, \dots, 0, j_0, \dots, j_{n-k-2}})_{j_{n-k-1}} \\
& \cong \mathbf{Hom}_{\mathbf{sSpace}_n}(\Delta[j_{n-k-1}, \dots, j_0, 0, \dots, 0], X^{F_n^k(\Delta[i_k, \dots, i_0])}) \\
& \cong \mathbf{Hom}_{\mathbf{sSpace}_n}(F_n^k(\Delta[i_k, \dots, i_0]) \times \Delta[j_{n-k-1}, \dots, j_0, 0, \dots, 0], X) \\
& \cong \mathbf{Hom}_{\mathbf{sSpace}_n}(\Delta[0, \dots, 0, i_k, \dots, i_0] \times \Delta[j_{n-k-1}, \dots, j_0, 0, \dots, 0], X) \\
& \cong \mathbf{Hom}_{\mathbf{sSpace}_n}(\Delta[j_{n-k-1}, \dots, j_0, i_k, \dots, i_0], X) \cong (X_{i_0, \dots, i_k, j_0, \dots, j_{n-k-2}})_{j_{n-k-1}}
\end{aligned}$$

which completes the proof.  $\square$

The case of  $k = 0$  and  $n = 1$  is discussed in [42, pg. 6]. More generally, the case  $k = 0$  and  $n \geq 1$  is discussed in [41, pg. 6]. Note that  $\mathbf{Map}_n^0$  defines an enrichment of  $\mathbf{sSpace}_n$  in simplicial sets.

Finally, we need another means to embed lower-dimensional simplicial spaces into higher-dimensional ones, by producing a space that is levelwise constant:

**Definition 2.9.** Let  $0 \leq k < n$ . Define the map

$$\iota_n^k : \mathbf{sSpace}_k \rightarrow \mathbf{sSpace}_n$$

such that, for any  $K \in \mathbf{sSpace}_k$  and  $i_0, \dots, i_n \geq 0$ ,

$$(\iota_n^k(K)_{i_0, \dots, i_{n-1}})_{i_n} = (K_{i_{n-k}, \dots, i_{n-1}})_{i_n}.$$

More formally, this is given by the precomposition map on presheaf categories induced by the functor  $\Delta^{n+1} \rightarrow \Delta^{k+1}$  sending

$$([i_n], \dots, [i_0]) \mapsto ([i_n], \dots, [i_{n-k}]).$$

**Notation 2.5.** For  $0 \leq k < n$ , write  $-\square_n^k-$  :  $\mathbf{sSpace}_n \times \mathbf{sSpace}_k \rightarrow \mathbf{sSpace}_n$  for the functor defined such that, for  $X \in \mathbf{sSpace}_n$  and  $K \in \mathbf{sSpace}_k$ ,

$$X \square_n^k K := X \times \iota_n^k(K).$$

If  $k = n$ , say that  $\square_n^k := \times$  is the Cartesian product on  $\mathbf{sSpace}_n$ .

For  $k = 0$ , this agrees with the  $\square$ -product given in [18, pg. 370].

The following two results above can be proven levelwise by hom-tensor adjunction:

**Proposition 2.7.** Let  $0 \leq k \leq n$ . Suppose  $K, X, Y \in \mathbf{sSpace}_n$ . There is a natural isomorphism

$$\mathbf{Map}_n^k(X \times K, Y) \cong \mathbf{Map}_n^k(X, Y^K).$$

**Proposition 2.8.** Let  $0 \leq m \leq k \leq n$ . Suppose  $K \in \mathbf{sSpace}_k$  and  $X, Y \in \mathbf{sSpace}_n$ . Then there is a natural isomorphism

$$\mathbf{Map}_n^m(X \square_n^k K, Y) \cong \mathbf{Map}_k^m(K, \mathbf{Map}_n^k(X, Y)).$$



## 2.2 Reedy Model Structures

Central to our chosen definition of  $(\infty, 2)$ -category is the construction of a suitable model structure. Our intention is to encode  $(\infty, 2)$ -categories as the fibrant objects in a model structure on  $\mathbf{ssSpace}$ .

A model structure is explicitly described by Bergner and Rezk in [8] and [9] which we will employ here. It is a natural extension of the model structure Rezk defined in [42] for complete Segal spaces. The approach is to take a left Bousfield localization of a pre-existing model structure that automatically exists on  $\mathbf{ssSpace}$ , whose weak equivalences are levelwise weak equivalences in  $\mathbf{sSet}$ . Taking such a starting point sets the stage for a truly ‘homotopy-theoretic’ approach, inheriting structure from  $\mathbf{sSet}$  to describe  $\infty$ -categorical phenomena in a manner derivative from the homotopy hypothesis. We will discuss this more later.

This initial model structure is known more commonly as the *Reedy model structure*. The most general case is studied in-depth in [24], but we will need only the specific cases of simplicial and bisimplicial spaces.

We begin with a few definitions with regards to Reedy categories in general, then specializing to the cases of interest to this discussion:

**Definition 2.10** ([24, Def. 15.1.2]). *A Reedy category  $\mathcal{C}$  is a small category equipped with two wide subcategories  $\mathcal{C}^+$  and  $\mathcal{C}^-$  called the direct and inverse subcategories and a function  $\mathbf{deg} : \mathbf{ob}(\mathcal{C}) \rightarrow \mathbb{Z}_{\geq 0}$  called the degree function<sup>3</sup>, such that:*

1. *Every morphism  $f$  in  $\mathcal{C}$  admits a unique factorization  $f = \alpha^+ \circ \alpha^-$ , where  $\alpha^+ \in \mathcal{C}^+$  and  $\alpha^- \in \mathcal{C}^-$ ;*
2. *For every  $\alpha^+ : c \rightarrow d$  in  $\mathcal{C}^+$  we have  $\mathbf{deg}(c) \leq \mathbf{deg}(d)$  and for every  $\alpha^- : a \rightarrow b$  in  $\mathcal{C}^-$  we have  $\mathbf{deg}(a) \geq \mathbf{deg}(b)$ . Equality holds in either case only if the morphism in question is an identity.*

**Proposition 2.9** ([24, Example 15.1.12]).  *$\Delta$  is a Reedy category, with degree map  $\mathbf{deg}([i]) = i$ ,  $\Delta^+$  the wide subcategory of injections and  $\Delta^-$  the wide subcategory of surjections.*

**Proposition 2.10.**  *$(\Delta^{op})^n$  is a Reedy category, with degree map  $\mathbf{deg}([i_1], \dots, [i_n]) = \sum_j i_j$ ,  $((\Delta^{op})^n)^+ = ((\Delta^-)^{op})^n$  and  $((\Delta^{op})^n)^- = ((\Delta^+)^{op})^n$ .*

*Proof.* Apply [24, Prop. 15.1.5] and [24, Prop. 15.1.6]. □

For any model category  $\mathcal{M}$  and any Reedy category  $C$ , the Reedy model structure is a particular model structure on the category  $\mathcal{M}^C$  whose weak equivalences are levelwise. We will not need the most general definition, to be found in [24, Def. 15.3.3], and will only discuss the instances we will make use of. We now turn to Rezk’s description in [42] of the Reedy model structure for simplicial spaces, where  $\mathcal{M} = \mathbf{sSet}$  and  $C = \Delta^{op}$ :

**Definition 2.11** ([42, pg. 6]). *A map  $f : X \rightarrow Y$  in  $\mathbf{sSpace}$  is a weak equivalence in the Reedy model structure if and only if it is such levelwise in  $\mathbf{sSet}$ . It is a Reedy fibration of simplicial spaces if and only if, for every  $n \geq 0$ , the map*

$$X_n \rightarrow Y_n \times_{\mathbf{Map}_1^0(\partial F_1^0(n), Y)} \mathbf{Map}_1^0(\partial F_1^0(n), X)$$

*is a fibration in  $\mathbf{sSet}$ , namely a Kan fibration.*

---

<sup>3</sup>Bergner and Rezk in [10] set the codomain of  $\mathbf{deg}$  to be  $\mathbb{N}$ . This is a trivial difference that will not affect any results.

It is the case that this produces a model structure by taking cofibrations to be maps with the left lifting property with respect to Reedy fibrations. It is possible to give a more elementary description using the results in [18]. We will not do so here.

We can now inductively make a similar statement about the Reedy model structure on  $n$ -uple simplicial spaces, in particular where  $\mathcal{M} = \mathbf{sSpace}_{n-1}$  and  $C = \Delta^{op}$ :

**Definition 2.12.** *Let  $f : X \rightarrow Y$  be a map in  $\mathbf{sSpace}_n$ . Then  $f$  is a weak equivalence in the Reedy model structure on  $\mathbf{sSpace}_n$  if and only if it is levelwise one in  $\mathbf{sSet}$  and a Reedy fibration of  $n$ -uple simplicial spaces if and only if, for every  $k \geq 0$ , the map*

$$X_k \rightarrow Y_k \times_{\mathbf{Map}_n^{n-1}(\partial F_n^0(k), Y)} \mathbf{Map}_n^{n-1}(\partial F_n^0(k), X)$$

*is a fibration in the Reedy model structure on  $\mathbf{sSpace}_{n-1}$ .*

A more elementary description not invoking Reedy fibrations of simplicial spaces is also possible, by instead taking  $\mathcal{M} = \mathbf{sSet}$  and  $C = (\Delta^{op})^n$ . It is shown by [24, Theorem 15.5.2] that these choices will yield identical model structures on  $\mathbf{sSpace}_n$ . We thus have:

**Proposition 2.11.** *Let  $n \geq 1$ . A map  $f : X \rightarrow Y$  is a Reedy fibration in  $\mathbf{sSpace}_n$  if and only if, for every  $i_1, \dots, i_n \geq 0$ , the map*

$$X_{i_1, \dots, i_n} \rightarrow Y_{i_1, \dots, i_n} \times_{\mathbf{Map}_n^0(\partial F_n^{n-1}(i_n, \dots, i_1), X)} \mathbf{Map}_n^0(\partial F(\Delta[i_n, \dots, i_1]), Y)$$

*is a Kan fibration.*

The proof is feasible by induction. For those interested in the technicalities, consider the case  $n = 2$ . If we take  $\Delta^{op}$  as our domain Reedy category, we have that  $\mathbf{Map}_1^0(\partial F_1^0(n), X)$  is a matching object of a simplicial space  $X$  for  $[n] \in \Delta^{op}$  and  $\mathbf{Map}_2^1(\partial F_2^0(m), Y)$  is as such for a bisimplicial space  $Y$  and  $[m] \in \Delta^{op}$ . Then, by [24, Lemma 15.5.1], we have for  $([m], [n]) \in (\Delta^{op})^2$  and a bisimplicial space  $X$ , the appropriate matching object is the pullback

$$\mathbf{Map}_1^0(\partial F_1^0(m), X_{n, \bullet}) \times_{\mathbf{Map}_1^0(\partial F_1^0(m), \mathbf{Map}_2^1(\partial F_2^0(n), X))} \mathbf{Map}_1^0(\partial F_1^0(n), X_{\bullet, m}).$$

This is shown by Proposition 2.3 to be the space  $\mathbf{Map}_2^1(\partial F_2^1(m, n), X)$ , as needed.

Of particular interest to us are *Reedy fibrant* objects, as a special case of these will be our  $(\infty, 1)$ -categories in the case of  $\mathbf{sSpace}$  and  $(\infty, 2)$ -categories for  $\mathbf{ssSpace}$ . In the former case, Rezk notes in [42, pg. 6] that the Reedy fibrant simplicial spaces are just those spaces  $X$  such that the map  $X_n \rightarrow \mathbf{Map}_1^0(\partial F_1^0(n), X)$  is a Kan fibration for all  $n \geq 0$ . Similarly, we find that a Reedy fibrant bisimplicial space  $X$  is one such that the map  $X_{n, m} \rightarrow \mathbf{Map}_2^0(\partial F_2^1(m, n), X)$  is a Kan fibration for all  $n, m \geq 0$ . In particular,  $X_0$  and  $X_{\bullet, 0}$  are both Reedy fibrant simplicial spaces. We alternatively might say that the maps  $X_n \rightarrow \mathbf{Map}_2^1(\partial F_2^0(n), X)$  are Reedy fibrations of simplicial spaces.

A useful fact about Reedy fibrations is that they are also levelwise Reedy fibrations:

**Proposition 2.12** ([18, pg. 366, Prop. 2.6]). *Reedy fibrations and cofibrations in  $\mathbf{sSpace}_n$  are also levelwise Reedy fibrations and cofibrations in  $\mathbf{sSpace}_{n-1}$ .*

In particular, every Reedy fibrant bisimplicial space is levelwise a Reedy fibrant simplicial space, which is then levelwise a Kan complex.

Bergner and Rezk in [10] demonstrate that  $\Delta$  is in fact an *elegant Reedy category*:

**Definition 2.13** ([10, Def. 2]). Let  $\mathcal{C}$  be a Reedy category and  $X : \mathcal{C}^{op} \rightarrow \mathbf{Set}$  a presheaf. Let  $c \in \mathcal{C}$  and  $x \in X(c)$ . Then  $x$  is degenerate if and only if there exists some non-identity map  $\alpha : c \rightarrow d$  in  $\mathcal{C}^-$  such that  $x = X(\alpha)(y)$  for some  $y \in X(d)$ .  $x$  is likewise nondegenerate if it is not degenerate.

**Definition 2.14** ([10, Def. 4]). An elegant Reedy category  $\mathcal{C}$  is a Reedy category such that, for every presheaf  $X : \mathcal{C}^{op} \rightarrow \mathbf{Set}$ , every  $c \in \mathcal{C}$  and  $x \in X(c)$ , there exists a unique map  $\alpha : c \rightarrow d$  in  $\mathcal{C}^-$  and unique nondegenerate  $y \in X(d)$  such that  $x = X(\alpha)(y)$ .

**Proposition 2.13** ([10, Cor. 4.4]).  $\Delta$  is an elegant Reedy category.

A consequence of this fact, stated as [10, Prop. 3.10], shows that the Reedy model structures on  $\mathbf{sSpace}$  and  $\mathbf{SSpace}$  are both just the *injective model structures*, where weak equivalences and cofibrations are levelwise:

**Proposition 2.14** ([10, Prop. 3.10]). Let  $\mathcal{C}$  be an elegant Reedy category and  $\mathcal{D}$  a category. Let  $\mathcal{M} := \mathbf{sSet}^{\mathcal{D}^{op}}$  be the model category of simplicial presheaves endowed with an injective model structure, so weak equivalences and cofibrations are levelwise. Then the injective and Reedy model structures (see [24, Thm. 15.3.4]) on  $\mathcal{M}^{\mathcal{C}}$  coincide.

We will not need the most general form of a Reedy model structure; the cases we care for have already been defined, setting  $\mathcal{D} = *$  or  $\mathcal{D} = \Delta$ . What is important is the following consequence:

**Corollary 2.1.1.** A map  $f : X \rightarrow Y$  in  $\mathbf{sSpace}_n$  is a Reedy cofibration if and only if  $(f_{i_0, \dots, i_{n-1}})_{i_n} : (X_{i_0, \dots, i_{n-1}})_{i_n} \rightarrow (Y_{i_0, \dots, i_{n-1}})_{i_n}$  is an inclusion of sets for all  $i_0, \dots, i_n \geq 0$ .

*Proof.* A cofibration in an injective model structure is a levelwise cofibration. Inductively set  $\mathcal{M} = \mathbf{sSet}$ ,  $\mathcal{D} = \Delta^k$  and  $\mathcal{C} = \Delta$  for  $0 \leq k < n$  to obtain that a Reedy cofibration in  $\mathbf{sSpace}_n$  is levelwise one in  $\mathbf{sSet}$ . Since cofibrations in  $\mathbf{sSet}$  are levelwise inclusions [26, Prop. 3.2.2], this completes the proof.  $\square$

**Corollary 2.1.2.** Every object in  $\mathbf{sSpace}_n$  is Reedy cofibrant.

Another result we will need is with regards to  $F_n^k$ :

**Proposition 2.15.** Let  $f : U \rightarrow V$  be a cofibration in  $\mathbf{sSpace}_k$  with  $k < n$ . Then  $F_n^k(f)$  is a cofibration in  $\mathbf{sSpace}_n$ .

*Proof.* By Corollary 2.1.1, it will suffice to prove that  $(F_n^k(f))_{i_0, \dots, i_{n-1}}_{i_n}$  is an inclusion of sets for every  $i_0, \dots, i_n \geq 0$ . This is just the map  $f_{i_1, \dots, i_{k+1}} : (U_{i_1, \dots, i_k})_{i_{k+1}} \rightarrow (V_{i_1, \dots, i_k})_{i_{k+1}}$ . However, since  $f$  is a cofibration, the map  $f_{i_1, \dots, i_k} : U_{i_1, \dots, i_k} \rightarrow V_{i_1, \dots, i_k}$  is a cofibration in  $\mathbf{sSet}$ , which means it must be a levelwise inclusion [26, Prop. 3.2.2]. This completes the proof.  $\square$

Note however that *trivial* cofibrations are not necessarily preserved by  $F_n^k$ . For instance,  $g : \Delta[1] \sqcup_{\Delta[0]} \Delta[1] \hookrightarrow \Delta[2]$  is a trivial cofibration in  $\mathbf{sSet}$ , but  $F_1^0(g)$  is not - indeed, we find that  $F_1^0(g)_1$  is a map from a two-element discrete simplicial set, the 1-simplices of the domain, to a three-element discrete simplicial set. This cannot possibly be a weak equivalence, so  $F_1^0(g)$  is not a levelwise weak equivalence.

One should note however that  $\iota_n^k$  does preserve both cofibrations and trivial cofibrations, as it simply produces many copies of the same trivial cofibration levelwise. We will not need this result.

Finally, we should say that one may take the Reedy model structure in a more general context, if the category  $\mathbf{sSpace}_k$  has a finer model structure imposed upon it:

**Definition 2.15.** Let  $\mathcal{M}$  be a model category whose underlying category is  $\mathbf{sSpace}_k$ . A weak equivalence in the Reedy model structure on  $\mathcal{M}^{\Delta^{op}}$  is taken levelwise, while a Reedy fibration in the Reedy model structure on  $\mathcal{M}^{\Delta^{op}}$  is a map  $f : A \rightarrow B$  in  $\mathcal{M}^{\Delta^{op}}$  such that for all  $n \geq 0$ , the induced map

$$A_n \rightarrow B_n \times_{\mathbf{Map}_2^1(\partial F_2^0(n), B)} \mathbf{Map}_2^1(\partial F_2^0(n), A)$$

is a fibration in  $\mathcal{M}$ .

## 2.3 Mapping Spaces and (Co)Fibrations

The interactions between our Reedy model structures and the various mapping spaces will be needed. We can infer what we need from a key structure called a *monoidal model structure*. We need a few definitions before we can proceed, following [26]:

**Definition 2.16** ([26, Def. 4.1.12]). Suppose  $\mathcal{C}$ ,  $\mathcal{D}$  and  $\mathcal{E}$  are categories. An adjunction of two variables is a 5-tuple of the form

$$(\otimes, \mathbf{Hom}_l, \mathbf{Hom}_r, \phi_r, \phi_l)$$

where  $\otimes : \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E}$ ,  $\mathbf{Hom}_l : \mathcal{C}^{op} \times \mathcal{E} \rightarrow \mathcal{D}$  and  $\mathbf{Hom}_r : \mathcal{D}^{op} \times \mathcal{E} \rightarrow \mathcal{C}$  are functors, while  $\phi_r$  and  $\phi_l$  are natural isomorphisms of the form

$$\mathbf{Hom}_{\mathcal{D}}(D, \mathbf{Hom}_l(C, E)) \xrightarrow{\phi_l} \mathbf{Hom}_{\mathcal{E}}(C \otimes D, E) \xrightarrow{\phi_r} \mathbf{Hom}_{\mathcal{C}}(C, \mathbf{Hom}_r(D, E))$$

Note that a Cartesian closed structure on a category  $\mathcal{C}$  induces an adjunction of two variables, by setting  $\mathcal{D} = \mathcal{E} = \mathcal{C}$ , defining  $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  to be the Cartesian product and  $\mathbf{Hom}_l = \mathbf{Hom}_r$  to be the exponential.

**Proposition 2.16.** Let  $0 \leq k \leq n$ . Then setting  $\mathcal{D} = \mathbf{sSpace}_k$  and  $\mathcal{C} = \mathcal{E} = \mathbf{sSpace}_n$  defines a Quillen adjunction of two variables with  $\otimes = \square_n^k$ ,  $\mathbf{Hom}_r = (-)^{l_n^k(-)}$  and  $\mathbf{Hom}_l = \mathbf{Map}_n^k$ .

*Proof.* If  $k = n$  then this is just the Cartesian closed structure, as mentioned above. If not, consider that we seek natural isomorphisms  $\phi_l$  and  $\phi_r$  of the form

$$\mathbf{Hom}_{\mathbf{sSpace}_k}(K, \mathbf{Map}_n^k(X, Y)) \xrightarrow{\phi_l} \mathbf{Hom}_{\mathbf{sSpace}_n}(X \square_n^k K, Y) \xrightarrow{\phi_r} \mathbf{Hom}_{\mathbf{sSpace}_n}(X, Y^{l_n^k(K)})$$

The isomorphism  $\phi_l$  is just the usual Cartesian hom-tensor adjunction whiskered with  $l_n^k$ . The morphism  $\phi_r$  is level 0 of the natural isomorphism in Proposition 2.8 with  $m = 0$ .  $\square$

It is then possible to ensure that these structures are compatible with a model structure in a useful manner for our purposes:

**Definition 2.17** ([26, Def. 4.2.1]). A Quillen adjunction of two variables is an adjunction of two variables  $(\otimes, \mathbf{Hom}_l, \mathbf{Hom}_r, \phi_l, \phi_r)$  over model categories  $\mathcal{C}, \mathcal{D}$  and  $\mathcal{E}$  such that, given a cofibration  $f : U \rightarrow V$  in  $\mathcal{C}$  and a cofibration  $g : W \rightarrow X$  in  $\mathcal{D}$ , the pushout product of  $f$  and  $g$

$$f \square g : P(f, g) = (V \otimes W) \sqcup_{U \otimes W} (U \otimes X) \rightarrow V \otimes X$$

is a cofibration, and is moreover trivial if either  $f$  or  $g$  is. We refer to  $\otimes$  as a Quillen bifunctor if  $\mathcal{C}, \mathcal{D}, \mathcal{E}, \mathbf{Hom}_l$  and  $\mathbf{Hom}_r$  are evident.

**Proposition 2.17.** *The adjunction of two variables in Proposition 2.16 is Quillen.*

*Proof.* Suppose  $f : U \rightarrow V$  is a cofibration in  $\mathbf{sSpace}_n$  and  $g : W \rightarrow X$  is a cofibration in  $\mathbf{sSpace}_k$ . By Corollary 2.1.1, these are both simply levelwise cofibrations in  $\mathbf{sSet}$ , so checking that the pushout product  $f \square g : (V \square_n^k W) \sqcup_{U \square_n^k W} (U \square_n^k W) \rightarrow (V \square_n^k X)$  is a cofibration amounts to checking this levelwise. This problem reduces to checking that for every  $i_1, \dots, i_n \geq 0$ , the map

$$(V_{i_1, \dots, i_n} \times W_{i_{n-k+1}, \dots, i_n}) \sqcup_{U_{i_1, \dots, i_n} \times W_{i_{n-k+1}, \dots, i_n}} (U_{i_1, \dots, i_n} \times X_{i_{n-k+1}, \dots, i_n}) \rightarrow (V_{i_1, \dots, i_n} \times X_{i_{n-k+1}, \dots, i_n})$$

is a cofibration in  $\mathbf{sSet}$ . However, the Cartesian product in  $\mathbf{sSet}$  is a Quillen bifunctor by [26, Prop. 4.2.8]. Hence,  $f \square g$  is a cofibration as desired. The above reasoning extends immediately to the case where  $f$  or  $g$  is a trivial cofibration, as these are again just levelwise trivial cofibrations in  $\mathbf{sSet}$ .  $\square$

The upshot of these technicalities is the following result, which we will make extensive use of:

**Lemma 2.2** ([26, Lemma 4.2.2]). *Suppose  $(\otimes, \mathbf{Hom}_l, \mathbf{Hom}_r, \phi_l, \phi_r)$  is an adjunction of two variables over model categories  $\mathcal{C}, \mathcal{D}$  and  $\mathcal{E}$ . Then the following conditions are equivalent:*

1. *The adjunction is Quillen.*
2. *Given a cofibration  $g : W \rightarrow X$  in  $\mathcal{D}$  and a fibration  $p : Y \rightarrow Z$  in  $\mathcal{E}$ , the induced map*

$$\mathbf{Hom}_{r, \square}(g, p) : \mathbf{Hom}_r(X, Y) \rightarrow \mathbf{Hom}_r(W, Y) \times_{\mathbf{Hom}_r(W, Z)} \mathbf{Hom}_r(X, Z)$$

*is a fibration, which is trivial if either  $g$  or  $p$  is.*

3. *Given a cofibration  $f : U \rightarrow V$  in  $\mathcal{C}$  and a fibration  $p : Y \rightarrow Z$  in  $\mathcal{E}$ , the induced map*

$$\mathbf{Hom}_{\square, l}(f, p) : \mathbf{Hom}_l(V, Y) \rightarrow \mathbf{Hom}_l(U, Y) \times_{\mathbf{Hom}_l(V, Z)} \mathbf{Hom}_l(V, Z)$$

*is a fibration, which is trivial if either  $f$  or  $p$  is.*

**Corollary 2.2.1.** *Suppose  $f : U \rightarrow V$  is a Reedy cofibration in  $\mathbf{sSpace}_n$  and  $p : Y \rightarrow Z$  is a Reedy fibration in  $\mathbf{sSpace}_n$ . Then the induced map*

$$\mathbf{Map}_n^k(f, p) : \mathbf{Map}_n^k(V, Y) \rightarrow \mathbf{Map}_n^k(U, Y) \times_{\mathbf{Map}_n^k(U, Z)} \mathbf{Map}_n^k(V, Z)$$

*is a Reedy fibration in  $\mathbf{sSpace}_k$ , which is trivial if either  $f$  or  $p$  is.*

**Corollary 2.2.2.** *If  $Y$  is Reedy fibrant in  $\mathbf{sSpace}_n$  and  $f : U \rightarrow V$  is a (trivial) Reedy cofibration in  $\mathbf{sSpace}_n$ , then the map*

$$\mathbf{Map}_n^k(f, Y) : \mathbf{Map}_n^k(V, Y) \rightarrow \mathbf{Map}_n^k(U, Y)$$

*is a (trivial) Reedy fibration in  $\mathbf{sSpace}_k$ .*

We note that this fact is not necessarily new in all cases. For instance, the case  $k = 0$  is well-known:

**Proposition 2.18.** *The mapping spaces  $\mathbf{Map}_n^0$  are precisely those obtained from the natural simplicial model structure on  $\mathbf{sSpace}_n$  described in [18, pg. 370, Cor. 2.13].*

## 2.4 Localizations

A few words should be said as well about left Bousfield localizations and how they will interact with injective model structures. We first need an auxiliary definition:

**Definition 2.18** ([24, Def. 16.1.2]). *Let  $\mathcal{M}$  be a model category. Let  $X \in \mathcal{M}$ . Define  $\mathbf{cc}_*(X) \in \mathcal{M}^\Delta$  to be the constant functor to  $X$ .*

*A cosimplicial resolution of  $X$  is a cofibrant approximation  $\tilde{X} \rightarrow \mathbf{cc}_*(X)$  in the Reedy model category structure on  $\mathcal{M}^\Delta$  [24, Def. 15.3.3]<sup>4</sup>.*

We are not too interested in the precise details of what such a Reedy model structure more generally looks like. For us, two facts are important:

**Proposition 2.19** ([24, Prop. 16.1.3]). *Let  $\mathcal{M}$  be a simplicial model category. If  $X$  is an object of  $\mathcal{M}$  and  $W \rightarrow X$  is a cofibrant approximation to  $X$ , then the cosimplicial object  $\tilde{W}$ , where  $\tilde{W}^n := W \otimes \Delta[n]$ , is a cosimplicial resolution of  $X$ .*

The place where this structure matters to us is in consideration of *homotopy function complexes*, which are heavily connected to homotopies between morphisms:

**Definition 2.19** ([24, Def. 17.1.1 and Notation 16.4.1]). *Let  $\mathcal{M}$  be a model category and  $X, Y \in \mathcal{M}$ . A left homotopy function complex from  $X$  to  $Y$  is a triple*

$$(\tilde{X}, \hat{Y}, \mathcal{M}(\tilde{X}, \hat{Y}))$$

*where  $\tilde{X}$  is a cosimplicial resolution of  $X$ ,  $\hat{Y}$  is a fibrant approximation to  $Y$  and  $\mathcal{M}(\tilde{X}, \hat{Y})$  is the simplicial set which at level  $n$  is the set  $\mathbf{Hom}_{\mathcal{M}}(\tilde{X}_n, \hat{Y})$ .*

For our purposes, left homotopy function complexes are sufficient. There are dual right homotopy function complexes and two-sided complexes [24, Def. 17.4.1], which involve simplicial resolutions [24, Def. 16.1.2]. We will refer to left homotopy function complexes as simply *homotopy function complexes*, as the left nature of them is unimportant.

**Corollary 2.2.3.** *Let  $X, Y \in \mathbf{sSpace}_n$  both be Reedy fibrant. Then  $\mathbf{Map}_n^0(X, Y)$  is a left homotopy function complex for  $X$  and  $Y$  in the Reedy model structure on  $\mathbf{sSpace}_n$ .*

*Proof.* Apply the cosimplicial resolution in Proposition 2.19, using  $X$  as a cofibrant approximation for itself by Corollary 2.1.2.  $\square$

We will use this as our homotopy function complex and may write it as  $\mathbf{map}(X, Y)$  when doing so.

**Definition 2.20** ([24, Definition 3.1.4]). *Let  $\mathcal{M}$  be a model category and  $C$  be a class of maps in  $\mathcal{M}$ .*

*An object  $W$  of  $\mathcal{M}$  is  $C$ -local if  $W$  is fibrant and for every element  $f : A \rightarrow B$  of  $C$ , the induced map of homotopy function complexes<sup>5</sup>*

$$f^* : \mathbf{map}(B, W) \rightarrow \mathbf{map}(A, W)$$

---

<sup>4</sup>Note that the source Reedy category is now  $\Delta$  rather than  $\Delta^{op}$ . We will not consider the ramifications of this difference further, as its application here is a technicality.

<sup>5</sup>Really, we should be using *functorial (left) homotopy function complexes*, as defined in [24, Def. 17.5.2], to ensure such maps are well-defined. Our implementations of homotopy function complexes will always be functorial, so we do not worry about this here.

is a weak equivalence. In turn, a map  $g : X \rightarrow Y$  in  $\mathcal{M}$  is a  $C$ -local equivalence if and only if for every  $C$ -local object  $W$ , the induced map

$$g^* : \mathbf{map}(Y, W) \rightarrow \mathbf{map}(X, W)$$

is a weak equivalence.

**Definition 2.21** ([24, Definition 3.3.1]). Let  $\mathcal{M}$  be a model category and  $C$  be a class of maps in  $\mathcal{M}$ .

The left Bousfield localization of  $\mathcal{M}$  with respect to  $C$ , if it exists, is a model category structure  $L_C\mathcal{M}$  on the underlying category of  $\mathcal{M}$  such that:

1. The weak equivalences of  $L_C\mathcal{M}$  are the  $C$ -local equivalences of  $\mathcal{M}$ ;
2. The cofibrations of  $L_C\mathcal{M}$  are the cofibrations of  $\mathcal{M}$ .

The fibrations are then induced.

Note then that a left Bousfield localization of a model structure will have the same cofibrations and trivial fibrations as before, while adding more weak equivalences and trivial cofibrations and removing some fibrations [24, Prop. 3.3.3]. Hence, in our eventual model structures defining  $(\infty, 2)$ -categories and  $(\infty, 1)$ -categories, levelwise weak equivalences will still be weak equivalences. Note also that fibrations in the localization will be Reedy fibrations, but the other way around will no longer be the case in general.

Existence of left Bousfield localizations will be guaranteed inductively by the following results acting in tandem:

**Proposition 2.20** ([24, Theorem 13.1.11] [24, Proposition 12.1.4]).  $\mathbf{sSet}$  is left proper and cellular.

**Proposition 2.21.** The Reedy model structure on  $\mathbf{sSpace}_n$  is left proper and cellular.

*Proof.* These model structures can be seen to be Reedy on the underlying model category  $\mathbf{sSet}$ , which is left proper. By [24, Theorem 15.3.4(2)], this means the Reedy model structure on  $\mathbf{sSpace}_n$  is left proper. A similar story holds for cellularity using [24, Theorem 15.7.6].  $\square$

**Proposition 2.22.** Let  $\mathcal{M}$  be a left proper and cellular model category and let  $C$  be a class of maps in  $\mathcal{M}$ . Then  $L_C\mathcal{M}$  exists and is moreover left proper and cellular.

It will be of great use to us to construct our sets of maps to localize by in the following data structure:

**Definition 2.22** ([27, pg. 54]). A presentation  $(C, M)$  consists of a small category  $C$  and a set of maps  $M$  in  $\mathbf{sSet}^C$ .

We will only care for presentations of the form  $(\Delta^{op}, M)$  and  $((\Delta^{op})^2, N)$ . The upshot for us is the ability to construct instances of the latter from the former:

**Definition 2.23** ([27, pg. 55]). Let  $(C, M)$  and  $(D, N)$  be presentations. Define a new presentation  $(C, M) \boxtimes (D, N)$  whose underlying category is the product of categories  $C \times D$  and whose set of distinguished maps in  $\mathbf{sSet}^{C \times D}$  is the set

$$M \boxtimes N := \{m \boxtimes 1_{z(d)}\}_{(m,d) \in M \times D} \cup \{1_{y(c)} \boxtimes n\}_{(c,n) \in C \times N}$$

where  $\mathbf{y} : C \rightarrow \mathbf{Set}^{C^{op}}$  and  $\mathbf{z} : D \rightarrow \mathbf{Set}^{D^{op}}$  are the Yoneda embeddings and  $\boxtimes : \mathbf{sSet}^C \times \mathbf{sSet}^D \rightarrow \mathbf{sSet}^{C \times D}$  denotes the functor sending a pair of presheaves  $(A, B)$  to the presheaf  $(A \boxtimes B) : (x, y) \mapsto A(x) \times B(y)$ .

Let  $C$  be a Reedy category. For us, to localize with respect to a presentation  $(C, M)$  simply means to localize the Reedy model structure on  $\mathbf{sSet}^C$  with respect to the maps  $M$ , namely to take the model category  $L_M \mathbf{sSet}^C$ .

## 2.5 Complete Segal Spaces

Complete Segal spaces were first developed by Rezk as a model for  $(\infty, 1)$ -categories in [42]. Though his intention was more so to use complete Segal spaces as models for homotopy theories akin to model categories, it soon became apparent that his construction could be iterated to obtain a definition of  $(\infty, n)$ -category, called *complete  $n$ -fold Segal spaces*. These objects are studied in [32], [9] and [27]. For us, the case  $n = 2$  will suffice.

We draw from the intuitions given in [34]. Suppose we wished to construct a model of an  $(\infty, 1)$ -category  $X$ . Where do we start? A reasonable place would be the *homotopy hypothesis*, which tells us that Kan complexes are a suitable model for  $\infty$ -groupoids. Hence, we can at least represent the ‘underlying  $\infty$ -groupoid’  $X_0 \in \mathbf{sSet}$  of  $X$ , the result of stripping away all non-invertible higher morphisms.

How do we encode the non-invertible morphisms in  $X$ ? Consider that a morphism should be represented by an ‘ $\infty$ -functor’  $[1] \rightarrow X$ , from the poset category  $[1] = \{0 \rightarrow 1\}$  to  $X$ . Hence, we might imagine the ‘ $\infty$ -groupoid of such functors’ as another simplicial set  $X_1 \in \mathbf{sSet}$ . One might note that  $X_0$  could similarly be interpreted as the  $\infty$ -groupoid of functors  $[0] \rightarrow X$ , where  $[0]$  is the discrete category with one object.

We immediately consider there to be maps  $s, t : X_1 \rightarrow X_0$  obtained by ‘precomposing’ with the maps  $[0] \rightarrow [1]$ . These maps extract the source and target of a 1-morphism. Moreover, there is a map  $i : X_0 \rightarrow X_1$  from the projection  $[1] \rightarrow [0]$ , which we interpret as giving the identity map of an object.

How do we compose morphisms? This will be answered by considering a new ‘ $\infty$ -groupoid’  $X_2 \in \mathbf{sSet}$  of ‘functors’  $[2] \rightarrow X$ , where in general  $[n]$  is the poset category  $\{0 < \dots < n\}$  for  $n \geq 0$ . Such a functor identifies a chain of two morphisms  $x \xrightarrow{f} y \xrightarrow{g} z$  and a third morphism  $x \xrightarrow{g \circ f} z$  such that said maps commute up to some higher equivalence - indeed, a functor between higher categories need not strictly respect composition of morphisms. There are clearly two maps  $X_1 \rightarrow X_2$  given by inserting identities on the left or right, along with three maps  $X_2 \rightarrow X_1$  given by extracting  $f, g$  or  $g \circ f$ .

Being able to take compositions now reduces to demanding that the induced *Segal map*

$$\gamma_2 : X_2 \rightarrow X_1 \times_{t, X_0, s} X_1$$

is invertible. Then, we have a path  $X_1 \times_{t, X_0, s} X_1 \rightarrow X_2 \rightarrow X_1$  sending  $(f, g)$  to  $g \circ f$ .

Is invertibility such a good idea? Similarly to quasicategories, we will abstain from choosing a particular composite and instead ask that there is a ‘contractible space of options’ for composites of two maps. More formally, we will ask that  $\gamma_2$  is a *weak equivalence* of simplicial sets. Indeed, we will go further and assert it is a trivial fibration - if everything is indeed a Kan complex, one would have that for each pair of maps  $f, g \in X_1$  such that  $t(f) = s(g)$ , the preimage  $\gamma_2^{-1}((f, g))$  would be a contractible Kan complex as desired.

For weak associativity and such, we will include spaces  $X_n$  of ‘functors’  $[n] \rightarrow X$ , representing chains of length  $n$  and all possible unbiased composites. Maps between these spaces are given by functors  $[m] \rightarrow [n]$ . We will then demand that all the remaining Segal



maps

$$\gamma_n : X_n \rightarrow X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

are trivial fibrations. The result of this discussion is a functor  $X : \Delta^{op} \rightarrow \mathbf{sSet}$  known as a *Segal space*.

**Definition 2.24** ([42, pg. 11]). *A Segal space is a Reedy fibrant simplicial space  $X : \Delta^{op} \rightarrow \mathbf{sSet}$  such that the Segal maps,*

$$\gamma_n : X_n \rightarrow X_1 \times_{X_0} \cdots \times_{X_0} X_1,$$

*are weak equivalences for all  $n \geq 2$ .*

**Definition 2.25.** *Let  $\mathbf{SeSp}$  be the full subcategory of  $\mathbf{sSpace}$  whose objects are the Segal spaces.*

Note that, because a Segal space  $X$  is Reedy fibrant, the spaces  $X_0$  and  $X_1$  are Kan complexes and the maps  $s := X_{(0)} = X_{d_1^0}$  and  $t := X_{(1)} = X_{d_0^1}$  must both be Kan fibrations. Hence, as noted in [42, pg. 11], the space  $X_1 \times_{X_0} \cdots \times_{X_0} X_1$  is actually a homotopy limit, ie.

$$X_1 \times_{X_0} \cdots \times_{X_0} X_1 \cong X_1 \times_{X_0}^h \cdots \times_{X_0}^h X_1.$$

Note also that the maps  $\gamma_n$  are necessarily trivial fibrations [42, pg. 11], a fact we must introduce more notation to explain.

**Definition 2.26.** *For  $n \geq 1$  and  $k \geq 1$ , let*

$$Sp(n) := \Delta[1] \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} \Delta[1] \in \mathbf{sSet}$$

*and set  $g_n : Sp(n) \hookrightarrow \Delta[n]$  to be the inclusion  $\langle 0, 1 \rangle \sqcup_{\langle 1 \rangle} \cdots \sqcup_{\langle n-1 \rangle} \langle n-1, n \rangle$ . For  $n = 0$ , let  $Sp(0) = \Delta[0]$  and  $g_0 = 1_{\Delta[0]}$ .*

We can clearly see then that  $\gamma_n$  is just  $\mathbf{Map}_1^0(F_1^0(g_n), X)$ . In the case that  $n < 2$ ,  $\gamma_n$  is just an identity. This is clearly a cofibration by Proposition 2.15, so  $\gamma_n$  is a Reedy fibration by Corollary 2.2.2 as needed.

An example is now in order. One of the classic litmus tests for any new definition of higher category is whether it supports a notion of *fundamental higher groupoid* of a topological space. This structure should encode all the weak homotopy theory of the space in question. The means we present to accomplish this in the  $(\infty, 1)$  and later  $(\infty, 2)$  cases are by no means novel, but an explicit reference in the literature remains elusive.

**Definition 2.27.** *Let  $\mathbf{Sing}_{\mathbf{sS}} : \mathbf{Top} \rightarrow \mathbf{sSpace}$  be the functor sending any  $X \in \mathbf{Top}$  to the space  $\mathbf{Sing}_{\mathbf{sS}}(X)$ , defined levelwise such that*

$$\mathbf{Sing}_{\mathbf{sS}}(X)_n := \mathbf{Sing}(X^{\Delta_t[n]})$$

*with simplicial maps induced by precomposition.*

We find that

$$\mathbf{Sing}_{\mathbf{sS}}(X)_{n,m} \cong \mathbf{Hom}_{\mathbf{Top}}(\Delta_t[n] \times \Delta_t[m], X).$$

Again, by [35, pg. 41, Theorem 2], this has a left adjoint  $|\bullet|_{\mathbf{sS}}$  given by a certain coend. We do not take a great deal of interest in this adjoint, so we will not discuss it further.

It is prudent that we evaluate how the intuitions that led us to defining Segal spaces apply to this example. Given a space  $X$ , the Kan complex  $\mathbf{Sing}_{\mathbf{SS}}(X)_0$  is just  $\mathbf{Sing}(X)$ , which is indeed the prototypical example of an  $\infty$ -groupoid, namely the fundamental  $\infty$ -groupoid of a topological space. This will be the underlying  $\infty$ -groupoid of our  $(\infty, 1)$ -category.

The space  $\mathbf{Sing}_{\mathbf{SS}}(X)_1$  is then the fundamental  $\infty$ -groupoid of the space  $X^{\Delta_t[1]}$  of paths in  $X$ . Indeed, the 1-morphisms in our  $(\infty, 1)$ -category will be paths in our topological space. Looking at  $\mathbf{Sing}_{\mathbf{SS}}(X)_n$  reveals in general that the  $n$ -simplices will simply be the topological  $n$ -simplices in  $X$ .

Now, the Segal map on  $\mathbf{Sing}_{\mathbf{SS}}(X)$  is one which sends a 2-simplex  $\Delta_t[2] \rightarrow X$  to the restriction  $|Sp(2)| \rightarrow X$ . That the Segal map is a weak equivalence is immediate: there is a deformation retract  $\Delta_t[2] \rightarrow |Sp(2)|$  of the spine inclusion, which induces a deformation retract of the Segal map itself by precomposition. This means that, given a chain of two 1-morphisms in  $\mathbf{Sing}_{\mathbf{SS}}(X)_1$ , they may be composed by applying this deformation retract to obtain a new 1-morphism in  $\mathbf{Sing}_{\mathbf{SS}}(X)$ . One may in general compose a chain of length  $n$  by doing the same with  $\Delta_t[n]$  and  $|Sp(n)|$ .

One might expect some notion of a ‘mapping space’ in a definition of  $\infty$ -category, namely an  $\infty$ -groupoid of higher morphisms between two fixed objects. This is easily obtained with Segal spaces:

**Definition 2.28** ([6, pg. 10]). *Let  $X$  be a Segal space and  $x, y \in X_{0,0}$ . The mapping space between  $x$  and  $y$ , written here as  $X(x, y)$ , is defined to be the pullback*

$$\begin{array}{ccc} X(x, y) & \longrightarrow & X_1 \\ \downarrow & \lrcorner & \downarrow \\ \{(x, y)\} & \longrightarrow & X_0 \times X_0 \end{array}$$

in  $\mathbf{sSet}$ .

Note that this is once again a homotopy pullback [6, pg. 16]. Moreover, it is clearly fibrant, as fibrations are preserved under pullback and the discrete space  $\{(x, y)\}$  is necessarily fibrant.

From now on, we introduce the following notation:

**Notation 2.6.** *Let  $X \in \mathbf{sSpace}_k$ . Let  $x_1, \dots, x_n \in (X_{0, \dots, 0})_0$ . Then write  $(-)^{x_1, \dots, x_n} : (\mathbf{sSpace}_{k-1})_{(X_0)^n} \rightarrow \mathbf{sSpace}_{k-1}$  be the functor sending  $A \rightarrow (X_0)^n$  to the pullback*

$$A^{x_1, x_2, \dots, x_n} := \{(x_1, \dots, x_n)\} \times_{(X_0)^n} A.$$

For instance,  $X(a, b) = X_1^{a, b}$ . We will also start writing  $X(a_1, \dots, a_n) := X_1^{a_1, \dots, a_n}$ . Moreover, we have Segal maps on fibers

$$\gamma_n^{a_1, \dots, a_n} : X_n^{a_1, \dots, a_n} \rightarrow X(a_1, \dots, a_n).$$

We would like a convenient term for when this is possible:

**Definition 2.29.** *Consider a cospan  $A \rightarrow (X_0)^m \leftarrow B$  in  $\mathbf{sSpace}_k$ . A map  $f : A \rightarrow B$  is object-fibered with respect to this cospan if  $f$  commutes with the cospan. If the cospan is evident, simply say  $f$  is object-fibered.*

**Proposition 2.23.**  $(-)^{x_1, \dots, x_n}$  preserves fibrations and trivial fibrations.

*Proof.* Pullbacks preserve fibrations and trivial fibrations in general.  $\square$

For instance, the maps  $\gamma_n^{a_1, \dots, a_n}$  are pullbacks of trivial fibrations of the form

$$\begin{array}{ccc} X_n^{a_1, \dots, a_n} & \longrightarrow & X_n \\ \gamma_n^{a_1, \dots, a_n} \downarrow & \lrcorner & \downarrow \gamma_n \\ X(a_1, \dots, a_n) & \hookrightarrow & X_1 \times_{X_0} \dots \times_{X_0} X_1 \end{array}$$

Another useful construction with regards to  $\infty$ -categories, and indeed one which underpins the core result of this paper, is the capacity to ‘collapse’ such a higher category down to a mere category, with the same objects but with just path components of mapping spaces as its hom-sets. This is not at all difficult to achieve:

**Definition 2.30** ([11, Def. 1.9]). *Let  $X$  be a Segal space. The homotopy category  $h_1(X) \in \mathbf{Cat}$  is the category whose objects are the elements of the set  $(X_0)_0$ , whose hom-sets are of the form*

$$\mathbf{Hom}_{h_1(X)}(x, y) := \pi_0(X(x, y))$$

with identities given by the degeneracies and composition by applying  $\pi_0$  to the zig-zag diagram

$$X(x, y) \times X(y, z) \xrightarrow{\cong} X(x, y, z) \hookrightarrow X(x, z) \xleftarrow{\gamma_2^{x, z}} X_2^{x, z} \xrightarrow{X_{d_1}^{x, z}} X(x, z)$$

Because  $\gamma_2^{x, z}$  is a weak equivalence of simplicial sets, the map  $\pi_0(\gamma_2^{x, z})$  is a bijection. Hence, after applying  $\pi_0$ , we will have a single function from the leftmost object of the diagram to the rightmost.

Note that since the Segal maps  $\gamma_2^{x, z}$  are trivial fibrations, we have a lifting problem of the form

$$\begin{array}{ccc} & & X_2^{x, z} \longrightarrow X(x, z) \\ & \nearrow \mu_2^{x, z} & \downarrow \gamma_2^{x, z} \\ X(x, y) \times X(y, z) \hookrightarrow (X_1 \times_{X_0} X_1)^{x, z} & \longrightarrow & (X_1 \times_{X_0} X_1)^{x, z} \end{array}$$

which admits the solution  $\mu_2^{x, z}$ . Taking  $\pi_0$  of this chain of morphisms induces the same composition map as above. Rezk and Rasekh make a similar observation in [42] and [40] respectively, though instead note only that since  $\gamma_2^{x, z}$  is a trivial fibration, any element of  $(X_1 \times_{X_0} X_1)^{x, z}$  can be lifted to an element of  $X_2^{x, z}$ , with any two such liftings being in the same path component. All of these approaches yield the same category. We will make explicit use of the approach of solving the entire lifting problem when we come to discuss homotopy bicategories.

**Proposition 2.24** ([42, Prop. 5.4]). *Let  $X$  be a Segal space. Then  $h_1(X)$  is a category.*

One may also prove that any map  $X \rightarrow Y$  between Segal spaces induces a functor  $h_1(X) \rightarrow h_1(Y)$  in a functorial manner, due to commutativity with Segal maps and degeneracy maps. Hence, one obtains a functor

$$h_1 : \mathbf{SeSp} \rightarrow \mathbf{Cat}.$$

It is perhaps useful to consider what happens to our running example of a Segal space. Here,  $\Pi_1 : \mathbf{Top} \rightarrow \mathbf{Cat}$  is the *fundamental groupoid* functor, sending a topological space to the groupoid whose objects are points in  $X$  and whose morphisms are homotopy classes of paths relative to start and end points [36, ch. 2].

**Proposition 2.25.** *There is a natural isomorphism*

$$h_1 \circ \mathbf{Sing}_{sS} \cong \Pi_1.$$

*Proof.* Consider a topological space  $X$  and take the category  $C := h_1(\mathbf{Sing}_{sS}(X))$ . The objects of this category are given by the underlying set of  $\mathbf{Sing}_1(X)_0$ , which is just the underlying set of  $X$ . Hence, there is a clear bijection from the objects of  $C$  to those of  $\Pi_1(X)$ .

Now, let  $x, y \in X$ . We wish to find a natural bijection

$$\mathbf{Hom}_C(x, y) := \pi_0(\{(x, y)\} \times_{X^2} X^{\Delta_t[1]}) \rightarrow \mathbf{Hom}_{\Pi_1(X)}(x, y).$$

Note that two paths  $\Delta_t[1] \cong [0, 1] \rightarrow X$  from  $x$  to  $y$  are identified in  $\Pi_1(X)$  if and only if there is a path  $[0, 1] \rightarrow X^{\Delta_t[1]}$  between these paths that is constant on source and target. This happens if and only if they are in the same path component of  $\pi_0(\{(x, y)\} \times_{X^2} X^{\Delta_t[1]})$ , so there is a natural bijection between these sets sending  $[\gamma]$  to  $[\gamma]$  for any path  $\gamma$  from  $x$  to  $y$  in  $X$ .

That these bijections respect composition and identities is evident.  $\square$

One may construct a model structure for Segal spaces, though we will not explore how this is done in depth here:

**Theorem 2.3** ([9, Theorem 5.2]). *There is a model structure obtained as a left Bousfield localization of the Reedy model structure on  $\mathbf{sSpace}$ , which we will write as  $\mathbf{SeSp}$ , whose fibrant objects are the Segal spaces.*

In short, one obtains this structure by localizing with respect to the spine inclusions  $F_1^0(g_n)$  for all  $n \geq 2$ .

We are, however, not quite done with our definition of an  $(\infty, 1)$ -category. One final addition is necessary to ensure that the ‘homotopical’ data of  $\infty$ -groupoids at each level of a Segal space and the ‘categorical’ data, displayed for instance in the homotopy category, coincide neatly. Achieving this will make the equivalences between  $(\infty, 1)$ -categories far more reasonable and intuitive.

**Definition 2.31** ([42, pg. 14]). *Let  $X$  be a Segal space. Then  $X_{heq} \subseteq X_1 \in \mathbf{sSet}$ , the space of homotopy equivalences of  $X$ , is the subspace of path components of  $X_1$  whose points are mapped to isomorphisms in  $h_1(X)$ .*

It is clear that the degeneracy map  $X_0 \rightarrow X_1$  factors through  $X_{heq}$ , as identity maps are indeed isomorphisms.

**Definition 2.32** ([42, pg. 14]). A complete Segal space  $X$  is a Segal space such that the map  $X_0 \rightarrow X_{heq}$  is a weak equivalence.

**Notation 2.7.** Write  $\mathbf{CSSP}$  for the full subcategory of  $\mathbf{sSpace}$  whose objects are the complete Segal spaces.

Once again, it is worthwhile checking what happens to singular spaces.

**Proposition 2.26.** Let  $X \in \mathbf{Top}$ . Then  $\mathbf{Sing}_{\mathbf{sS}}(X)$  is a complete Segal space.

*Proof.* We know that  $h_1(\mathbf{Sing}_{\mathbf{sS}}(X)) \cong \Pi_1(X)$  is a groupoid. Hence,  $\mathbf{Sing}_{\mathbf{sS}}(X)_{heq} = \mathbf{Sing}_{\mathbf{sS}}(X)_1$ , so we need only show that the map

$$X^{\Delta_t[0]} \rightarrow X^{\Delta_t[1]}$$

induced by the degeneracy  $\Delta_t[1] \rightarrow \Delta_t[0]$  is a weak equivalence. This is so because the degeneracy map is a homotopy equivalence.  $\square$

One may tighten the model structure  $SeSp$  to obtain a sensible homotopy theory for complete Segal spaces. Moreover, the weak equivalences are now analogous to equivalences of categories:

**Definition 2.33** ([42, pg. 16]). A map  $f : X \rightarrow Y$  between Segal spaces is a Dwyer-Kan equivalence if and only if:

1. The induced functor  $h_1(f) : h_1(X) \rightarrow h_1(Y)$  is an equivalence of categories;
2. For every  $x, y \in X_{0,0}$ , the induced map  $X(x, y) \rightarrow Y(f(x), f(y))$  is a weak equivalence.

This is much more akin to the usual ‘fully faithful and essentially surjective’ definition of an equivalence of categories. The second part of the definition is the extension of ‘full faithfulness’ to weak equivalence, as the first property implies only a bijection on path components.

**Theorem 2.4** ([42, pg. 15, Theorem 7.2] [42, pg. 17, Theorem 7.7] [42, pg. 28]). There is a closed model structure obtained as a left Bousfield localization of the Reedy structure on  $\mathbf{sSpace}$ , which we will write as  $\mathbf{CSSP}$ , whose fibrant objects are the complete Segal spaces and whose weak equivalences between Segal spaces are the Dwyer-Kan equivalences.

The model structure is defined as a left Bousfield localization of the Reedy model structure on  $\mathbf{sSpace}$ . It is given by the following presentation:

**Definition 2.34** ([27, pg. 55-56]). The Rezk presentation  $(\Delta^{op}, S^{op})$  is given by the set  $S^{op}$ , consisting of the opposites of the elements of the set  $S$  of maps in  $\mathbf{sSet}^{\Delta^{op}} = \mathbf{sSpace}$  containing:

1. The **Segal maps**  $F_1^0(g_n) : F_1^0(Sp(n)) \hookrightarrow F_1^0(n)$ .
2. The **completeness map**

$$F_1^0(0) \sqcup_{F_1^0(\langle(0,0)\rangle), F_1^0(1), F_1^0(\langle(0,2)\rangle)} F_1^0(3) \sqcup_{F_1^0(\langle(1,3)\rangle), F_1^0(1), F_1^0(\langle(0,0)\rangle)} F_1^0(0) \rightarrow F_1^0(0)$$

given by the maps  $1_{F_1^0(0)}$  and  $F_1^0(\langle(0, 0, 0, 0)\rangle)$ .

It was shown by Rezk in [41, pg. 30] that locality with respect to the completeness map above induces precisely the completeness condition on Segal spaces.

## 2.6 Complete 2-fold Segal Spaces

We are now ready to begin our foray into the world of  $(\infty, 2)$ -categories. Our model of choice will be *complete 2-fold Segal spaces*, which will be built out of complete Segal spaces.

In order to facilitate the structure of an  $(\infty, 2)$ -category  $X$ , we should consider both horizontal and vertical composition. Hence, for each  $n, m \geq 0$ , we will now have an  $\infty$ -groupoid of ‘grids’ of 2-morphisms of horizontal length  $n$  and vertical length  $m$ , which we will write as  $X_{n,m}$ . We might see this as the  $\infty$ -groupoid of  $\infty$ -functors  $[n] \times [m] \rightarrow X$ .

This should, for a fixed  $n \geq 0$ , induce a complete Segal space  $X_{n,\bullet}$  where composition is vertical. In the special case  $n = 1$ , we have what we might call the  $(\infty, 1)$ -category of 1-morphisms  $X_{1,\bullet}$ . We then have  $(\infty, 1)$ -categories  $X_{\bullet,n}$  whose composition is instead horizontal.

A starting point to formalize this intuition is as follows:

**Definition 2.35** ([6, pg. 15]). *A double Segal space is a Reedy fibrant bisimplicial space such that the Segal maps,*

$$\begin{aligned}\gamma_{n,\bullet} : X_{n,\bullet} &\rightarrow X_{1,\bullet} \times_{X_{0,\bullet}} \cdots \times_{X_{0,\bullet}} X_{1,\bullet}, \\ \gamma_{\bullet,n} : X_{\bullet,n} &\rightarrow X_{\bullet,1} \times_{X_{\bullet,0}} \cdots \times_{X_{\bullet,0}} X_{\bullet,1},\end{aligned}$$

*are weak equivalences for all  $n \geq 2$ , so  $X_{k,\bullet}$  and  $X_{\bullet,k}$  are Segal spaces.*

This is in fact quite analogous to how we previously built complete Segal spaces from  $\infty$ -groupoids. At each level, a double Segal space  $X$  will be a complete Segal space. We could indeed choose to see  $X_0$  as the ‘underlying  $(\infty, 1)$ -category’ of  $X$ , containing only the invertible 2-morphisms. More generally,  $X_n$  will be a complete Segal space, which we could choose to interpret as the  $(\infty, 1)$ -category of  $(\infty, 2)$ -functors  $[n] \rightarrow X$ . Precomposition yields a functor from  $\Delta^{op}$  to **CSSP**, which is a Reedy fibrant bisimplicial space satisfying a Segal condition valued in  $(\infty, 1)$ -categories instead of  $\infty$ -groupoids. This can be made more precise:

**Proposition 2.27.** *A bisimplicial space  $X$  is a double Segal space if and only if it is Reedy fibrant, levelwise a Segal space and is such that the Segal maps for all  $n \geq 2$*

$$\gamma_n : X_n \rightarrow X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

*are weak equivalences in  $SeSp$ .*

*Proof.* The Segal map  $\gamma_n$  is a Reedy fibration. Thus, since Reedy trivial fibrations and trivial fibrations in  $SeSp$  are the same [24, Prop. 3.3.3], the maps  $\gamma_n$  are weak equivalence in  $SeSp$  if and only if they are weak equivalences levelwise. This means that for each  $m \geq 0$ , the map

$$\gamma_{n,m} : X_{n,m} \rightarrow X_{1,m} \times_{X_{0,m}} \cdots \times_{X_{0,m}} X_{1,m}$$

is a weak equivalence, which is precisely the first case of Segal maps for a double Segal space. The second case is given if and only if each  $X_n$  is a Segal space.  $\square$

We had to separately assert that  $X$  was levelwise a Segal space, since Reedy fibrancy alone will not guarantee it. One may note that this is precisely the notion of 2-fold Segal space in [7, Def. 3.4] minus ‘essential constancy’, a property we will return to later.

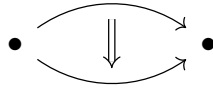
There is a stronger version of this statement in [6, Prop. 6.3]. This result is only to secure intuitions for us, so will not study it further here.

A useful construction with double Segal spaces will be an analogue of the mapping spaces we saw with Segal spaces:

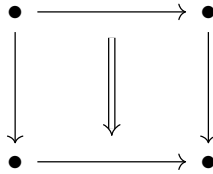
**Definition 2.36** ([6, pg. 16]). *Let  $X$  be a double Segal space. Let  $x, y \in X_{0,0,0}$ . Then the mapping space  $X(x, y)$  is defined to be the pullback in  $\mathbf{sSpace}$  of the form*

$$\begin{array}{ccc} X(x, y) & \longrightarrow & X_1 \\ \downarrow & \lrcorner & \downarrow \\ \{(x, y)\} & \longrightarrow & X_0 \times X_0. \end{array}$$

As noted in [6, pg. 15], double Segal spaces do not quite model  $(\infty, 2)$ -category theory as we may wish them to. To see the issue, as our prior intuitions have demonstrated, we should consider  $X_{1,1}$  to be the  $\infty$ -groupoid of 1-morphisms with 2-morphisms between them. We might consider the source and target maps  $X_{1,1} \rightarrow X_{1,0}$  to give the source and target 1-morphisms. However, there are now four possible source and target objects, given by the four maps  $X_{1,1} \rightarrow X_{0,0}$ . These do not have to equate to each other to give a ‘globular picture’ of 2-morphisms:



Instead, we have to contend with the ‘vertical 1-morphisms’ in  $X_{0,1}$  that we have neglected thus far. We instead have a ‘cubical’ picture of 2-morphisms:



More explicitly, given some  $f \in (X_{1,1})_0$ , we could identify the parts of this diagram as

$$\begin{array}{ccc} X_{(d_1^1, d_1^1)}(f) \in X_{0,0} & \xrightarrow{X_{(d_1^1, id)}(f) \in X_{1,0}} & X_{(d_1^1, d_0^1)}(f) \in X_{0,0} \\ \downarrow X_{(id, d_1^1)}(f) \in X_{0,1} & \Downarrow f \in X_{1,1} & \downarrow X_{(id, d_0^1)}(f) \in X_{0,1} \\ X_{(d_0^1, d_1^1)}(f) \in X_{0,0} & \xrightarrow{X_{(d_1^1, id)}(f) \in X_{1,0}} & X_{(d_0^1, d_0^1)}(f) \in X_{0,0} \end{array}$$

We take a greater interest in the globular picture here. Hence, the vertical spaces  $X_{0,\bullet}$  will have to be brushed under the rug somehow, which we will accomplish by demanding that they are *essentially constant*:

**Definition 2.37** ([11, pg. 12]). *A Segal space  $X$  is essentially constant if and only if the natural map  $q : \iota_1^0(X_0) \rightarrow X$ , defined levelwise such that  $q_n : X_0 \rightarrow X_n$  is induced by the degeneracy map  $\Delta[n] \rightarrow \Delta[0]$ , is a weak equivalence in  $SeSp$ .*

It will be beneficial for us to establish a running example of an  $(\infty, 2)$ -category. This will be done in much the same way as with complete Segal spaces:

**Definition 2.38.** *Suppose  $X \in \mathbf{Top}$ . Then define  $\mathbf{Sing}_{\mathbf{ssS}}(X)$  to be the bisimplicial space such that, for all  $n \geq 0$ ,*

$$\mathbf{Sing}_{\mathbf{ssS}}(X)_n := \mathbf{Sing}_{\mathbf{sS}}(X^{\Delta_t[n]}).$$

Note then that

$$(\mathbf{Sing}_{\mathbf{ssS}}(X)_{a,b})_c \cong \mathbf{Hom}_{\mathbf{Top}}(\Delta_t[a] \times \Delta_t[b] \times \Delta_t[c], X).$$

It is evident that this is a double Segal space. Indeed, we have that  $\mathbf{Sing}_{\mathbf{ssS}}(X)_{\bullet,k} \cong \mathbf{Sing}_{\mathbf{ssS}}(X)_{k,\bullet} = \mathbf{Sing}_{\mathbf{sS}}(X^{\Delta_t[k]})$ , which is in fact a complete Segal space.

As for our intuitions, the idea that  $\mathbf{Sing}_{\mathbf{ssS}}(X)_0$  should be seen as the underlying  $(\infty, 1)$ -category of  $\mathbf{Sing}_{\mathbf{ssS}}(X)$  is somewhat immediate by definition. Moreover, the interpretation of  $\mathbf{Sing}_{\mathbf{ssS}}(X)_1$  as the  $(\infty, 1)$ -category of morphisms also applies. The intuition carries on for higher levels.

Now, consider the cubical picture for 2-morphisms from before. Indeed, the space  $\mathbf{Sing}_{\mathbf{ssS}}(X)_{1,1}$  is quite literally  $\mathbf{Sing}(X^{\Delta_t[1] \times \Delta_t[1]})$ , the space of squares inside  $X$ . The face maps identify the edges and corners of this square. With the Segal maps, we can compose these squares either horizontally or vertically by aligning edges.

Note also however that the vertical maps are ‘essentially constant’; they are paths, which can be contracted so the square is essentially ‘pinched’ into a globular picture, up to homotopy. Hence, the vertical data is rather negligible. Of course, in this example the horizontal data is also as such since  $\mathbf{Sing}_{\mathbf{ssS}}(X)$  is really an  $\infty$ -groupoid in disguise, but more general examples of  $(\infty, 2)$ -categories are not so simple.

We will start our strengthening of double Segal spaces by first moving from levelwise Segal spaces to levelwise complete Segal spaces:

**Definition 2.39.** *A complete 2-uple Segal space is a Reedy fibrant functor  $X : \Delta^{op} \rightarrow \mathbf{sSpace}$  in the Reedy model structure on  $\mathbf{sSpace}$  such that  $X_{k,\bullet}$  and  $X_{\bullet,k}$  are both complete Segal spaces.*

It is clear that  $\mathbf{Sing}_{\mathbf{ssS}}(X)$  is a complete 2-uple Segal space for any  $X \in \mathbf{Top}$ , as  $\mathbf{Sing}_{\mathbf{sS}}(X^{\Delta_t[k]})$  is a complete Segal space for all  $k \geq 0$ .

We still need to strengthen this further. For us, a complete 2-fold Segal space will be equivalent to a Reedy fibrant rendition of the definition given in [27, pg. 11].

**Definition 2.40.** *A complete 2-fold Segal space  $X$  is a complete 2-uple Segal space such that  $X_0$  is essentially constant.*

**Proposition 2.28.** *There is a model category structure on  $\mathbf{ssSpace}$ , which we will title  $\mathbf{CSSP}_2$ , whose fibrant objects are precisely the complete 2-fold Segal spaces.*

Bergner in [6, Theorem 6.11] makes a similar statement, albeit with respect to a different definition of complete 2-fold Segal space and a different approach to defining a model structure. Our model structure, following [27, pg. 55-56] but this time in the Reedy fibrant case, is the left Bousfield localization of  $\mathbf{ssSpace}$  with regards to the following presentation:



**Definition 2.41.** *The 2-fold Lurie presentation is the presentation  $(\Delta, S) \boxtimes (\Delta, S)$  with one extra map:*

1. *The **essential constancy maps**, for each  $n \geq 0$ , defined as*

$$F_2^0(\langle 0, \dots, 0 \rangle, \langle 0 \rangle) : F_2^0(m, 0) \rightarrow F_2^0(0, 0).$$

This can be generalized to all  $\mathbf{sSpace}_n$  to obtain the *Lurie presentation* in [27, pg. 56]. The model structure does not particularly interest us here, other than one fact about it:

**Proposition 2.29.** *Let  $X$  be a complete 2-fold Segal space. Then the Segal maps are trivial fibrations in both *CSSP* and the Reedy model structure on  $\mathbf{sSpace}$ .*

*Proof.* We know that a complete 2-fold Segal space is a complete 2-uple Segal space, meaning it is implicitly a Reedy fibrant functor  $\Delta^{op} \rightarrow \mathbf{sSpace}$  with respect to the Reedy model structure on  $\mathbf{sSpace}$ . Thus, the Segal maps  $\gamma_k$  are Reedy fibrations of simplicial spaces by Corollary 2.2.2. We also know that each  $X_{\bullet, k}$  is a complete Segal space, meaning the maps  $\gamma_k$  are levelwise weak equivalences. Thus,  $\gamma_k$  is a Reedy trivial fibration, meaning it is as such in *CSSP*; indeed, by [24, Prop. 3.3.3], left Bousfield localizations of model structures have the same trivial fibrations.  $\square$

We henceforth employ the notation  $A^{a_1, \dots, a_n}$  and  $X(a_1, \dots, a_n)$  similarly to for simplicial spaces. Again, note that the fibers of the Segal maps  $\gamma_n^{a_1, \dots, a_n}$  are trivial fibrations in *SeSp* if  $X$  is a double Segal space and in *CSSP* if  $X$  is a complete 2-fold Segal space.

There is a clear example of a complete 2-fold Segal space, by all above comments:

**Proposition 2.30.** *Let  $X \in \mathbf{Top}$ . Then  $\mathbf{Sing}_{\mathbf{ssS}}(X)$  is a complete 2-fold Segal space.*

## 2.7 Composition and Composite Diagrams

Before we conclude on complete 2-fold Segal spaces, it will serve us well to think further on how composition in such an  $(\infty, 2)$ -category really works. Our intuitions for complete 2-fold Segal spaces and discussion of  $h_1$  show that, for a complete 2-fold Segal space  $X$ , constructing a binary composition operation amounts to solving the lifting problem

$$\begin{array}{ccc} & & X_2 \xrightarrow{X_{(0,2)}} X_1 \\ & \nearrow \mu_2 & \downarrow \gamma_2 \\ X_1 \times_{X_0} X_1 & \xrightarrow{id} & X_1 \times_{X_0} X_1 \end{array}$$

then fibering the map  $X_{(0,2)} \circ \mu_2$  over some pair  $\{(x, z)\} \in ((X_0 \times X_0)_0)_0$  and finally precomposing with the inclusion of  $X(x, y) \times X(y, z)$  for some  $y \in (X_{0,0})_0$  to obtain a map

$$\circ^{x,y,z} : X(x, y) \times X(y, z) \hookrightarrow (X_1 \times_{X_0} X_1 \times_{X_0} X_1)^{x,z} \xrightarrow{\mu_2^{x,z}} X_2^{x,z} \xrightarrow{X_{(0,2)}^{x,z}} X(x, z).$$

We should think of this as binary composition on the hom-spaces  $X(x, y)$  and  $X(y, z)$ . In general, we can take any sequence of objects  $x_0, \dots, x_n \in (X_{0,0})_0$  and solve a lifting

problem

$$\begin{array}{ccc}
& & X_n \xrightarrow{X_{(0,n)}} X_1 \\
& \nearrow \mu_n \text{ (dashed)} & \downarrow \gamma_n \\
X_1 \times_{X_0} \cdots \times_{X_0} X_1 & \xrightarrow{id} & X_1 \times_{X_0} \cdots \times_{X_0} X_1
\end{array}$$

to obtain a composition map

$$\circ^{x_0, \dots, x_n} : \prod_{i=1}^n X(x_{i-1}, x_i) \hookrightarrow (X_1 \times_{X_0} \cdots \times_{X_0} X_1)^{x_0, x_n} \xrightarrow{\mu_n^{x_0, x_n}} X_n^{x_0, x_n} \xrightarrow{X_{(0,n)}^{x_0, x_n}} X(x_0, x_n).$$

Note that  $\mu_n$  must be object-fibered with respect to the cospan

$$X_n \xrightarrow{X_{(0)} \times \cdots \times X_{(n)}} (X_0)^{n+1} \xleftarrow{(X_{(0)} \times X_{(1)}) \times_{1X_0} \cdots \times_{1X_0} (X_{(0)} \times X_{(1)})} X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

as defined in Definition 2.29.

**Proposition 2.31.** *Let  $X$  be a complete 2-fold Segal space. Consider a solution  $g$  to a lifting problem in CSSP*

$$\begin{array}{ccc}
A & \xrightarrow{s} & B \\
i \downarrow & \nearrow g \text{ (dashed)} & \downarrow p \\
C & \xrightarrow{t} & D
\end{array}$$

where every morphism besides  $g$  is object-fibered over some maps to  $(X_0)^{n+1}$ . Then  $g$  is object-fibered and for any  $x_0, \dots, x_n \in (X_{0,0})_0$ , the diagram<sup>6</sup>

$$\begin{array}{ccc}
A^{x_0, \dots, x_n} & \xrightarrow{s^{x_0, \dots, x_n}} & B^{x_0, \dots, x_n} \\
i^{x_0, \dots, x_n} \downarrow & \nearrow g^{x_0, \dots, x_n} \text{ (dashed)} & \downarrow p^{x_0, \dots, x_n} \\
C^{x_0, \dots, x_n} & \xrightarrow{t^{x_0, \dots, x_n}} & D^{x_0, \dots, x_n}
\end{array}$$

commutes.

*Proof.* Consider the chosen maps  $f_Z : Z \rightarrow (X_0)^{n+1}$  for  $Z \in \{A, B, C, D\}$ . Then

$$f_C = f_D \circ t = f_D \circ (p \circ g) = (f_D \circ p) \circ g = f_B \circ g$$

and so  $g$  is object-fibered. Commutativity then follows from functoriality of  $(-)^{x_0, \dots, x_n}$ .  $\square$

The map  $\circ^{x_0, \dots, x_n}$  is then ‘unbiased composition,’ taking a sequence of 1-morphisms in  $X$

$$x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} x_{n-1} \xrightarrow{f_n} x_n$$

and composing them into a single map  $f_n \circ \cdots \circ f_0$  at once. These operations amount to horizontal composition in our  $(\infty, 2)$ -category, while the Segal maps in each complete Segal space  $X_n$  induce vertical composition. We package this into a definition:

<sup>6</sup>Note that this may no longer be a lifting problem, as  $i^{x_0, \dots, x_n}$  may not be a (trivial) cofibration. For us, usually  $A = \emptyset$  and all objects are cofibrant when we want a lifting problem, so this doesn’t matter.

**Definition 2.42.** Let  $X$  be a complete 2-fold Segal space. Then a choice of horizontal compositions is a sequence of maps  $\mu_n : X_1 \times_{X_0} \cdots \times_{X_0} X_1 \rightarrow X_n$  for  $n \geq 2$  solving the lifting problems

$$\begin{array}{ccc} & & X_n \\ & \nearrow \mu_n & \downarrow \gamma_n \\ X_1 \times_{X_0} \cdots \times_{X_0} X_1 & \xrightarrow{id} & X_1 \times_{X_0} \cdots \times_{X_0} X_1 \end{array}$$

**Notation 2.8.** Given a choice of horizontal composition maps  $\mu_n$ , write  $\mu_1 = 1_{X_1}$  and  $\mu_0 = 1_{X_0}$ .

Recall that since  $\gamma_1$  and  $\gamma_0$  are just identities, the maps  $\mu_1$  and  $\mu_0$  are solutions of similar lifting problems. In fact, they are the only solutions of their respective lifting problems.

**Notation 2.9.** For a complete 2-fold Segal space  $X$ , given  $n > 0$  and  $x_0, \dots, x_n \in (X_{0,0})_0$ , along with a choice of horizontal compositions  $(\mu_n)_{n \geq 0}$ , write  $\circ^{x_0, \dots, x_n} : \prod_{i=1}^n X(x_{i-1}, x_i) \rightarrow X(x_0, x_n)$  for the map described above. If  $n = 0$ , write  $\circ^{x_0} = X_{s_0^1}^{x_0, x_0} : * \cong \{x_0\} \hookrightarrow X(x_0, x_0)$ .

Note we did not need to separately specify  $\circ^{x_0}$ , as if  $n = 0$  then  $\prod_{i=1}^n X(x_{i-1}, x_i) \cong *$ . We do so simply for clarity; in future, the  $n = 0$  case will be implicitly covered in this manner.

Note also that there is almost never a unique choice of horizontal compositions - there are a multitude of possible solutions to the lifting problems in general. We will see in time that these choices, while not equal, are in fact always equivalent in a way we will make precise later.

One might wonder how more complex ‘nested’ composition operations look, for instance composing a chain of 1-morphisms in  $X$  of the form  $v \xrightarrow{f} w \xrightarrow{g} x \xrightarrow{h} y \xrightarrow{k} z$  into the composite  $(k \circ h) \circ (g \circ f)$ . This is of course merely a matter of applying our operations  $\circ^{x_0, \dots, x_n}$  in sequence, yet a more direct interpretation is also possible, one which will be of immense importance in developing our homotopy bicategories.

We introduce some new notation for mapping spaces defined by a simplicial set:

**Notation 2.10.** Write  $X_\bullet : \mathbf{sSet}^{op} \rightarrow \mathbf{sSpace}$  for the functor  $\mathbf{Map}_2^1(F_2^0(\bullet), X)$ , sending

$$K \mapsto X_K := \mathbf{Map}_2^1(F_2^0(K), X).$$

It is clear then that if  $j : K \rightarrow L$  is a cofibration, the map  $j^* = X_j : X_L \rightarrow X_K$  is a Reedy fibration.

Note that everything we have written can be rephrased in terms of mapping spaces:

$$\begin{array}{ccc} & & X_n \xrightarrow{X_{(0,n)}} X_1 \\ & \nearrow \mu_n & \downarrow \gamma_n \\ X_{Sp(n)} & \xrightarrow{id} & X_{Sp(n)} \end{array}$$

The composition operation  $(- \circ -) \circ (- \circ -)$  can now be constructed from the solutions

to the sequence of two lifting problems

$$\begin{array}{ccc}
& & X_2 \xrightarrow{X_{(0,2)}} X_1 \\
& & \downarrow \gamma^2 \\
& & X_{Sp(2)} \\
& & \uparrow X_{(0,2) \times_{1_{X_0}} X_{(0,2)}} \\
& & X_{\Delta[2] \sqcup_{\Delta[0]} \Delta[2]} \\
& & \downarrow \gamma^2 \times_{1_{X_0}} \gamma^2 \\
X_{Sp(4)} & \xrightarrow{id} & X_{Sp(4)} \\
\uparrow \mu_2 \circ (X_{(0,2)} \times_{1_{X_0}} X_{(0,2)}) \circ (\mu_2 \times_{1_{X_0}} \mu_2) & & \uparrow \mu_2 \times_{1_{X_0}} \mu_2
\end{array}$$

The overall map is then, for  $v, w, x, y, z \in (X_{0,0})_0$ , the path

$$\begin{array}{ccc}
X(v, w) \times X(w, x) \times X(x, y) \times X(y, z) & \hookrightarrow & X_{Sp(4)}^{v,z} \\
& & \downarrow X_{(0,2)}^{v,z} \circ (\mu_2 \circ (X_{(0,2)} \times_{1_{X_0}} X_{(0,2)}) \circ (\mu_2 \times_{1_{X_0}} \mu_2))^{v,z} \\
& & X(v, z)
\end{array}$$

We can reinterpret this algebraic monstrosity in a more geometric light. Consider how the cospan

$$X_2 \xrightarrow{\gamma^2} X_{Sp(2)} \leftarrow X_{\Delta[2] \sqcup_{\Delta[0]} \Delta[2]}$$

appears in the above lifting problem. The two lifts can be combined into a single lift to the pullback of the cospan

$$X_{\Delta[2] \sqcup_{\Delta[0]} \Delta[2]} \times_{X_{Sp(2)}} X_2$$

which, since  $F_2^0$  preserves colimits and by Proposition 2.16, we can rephrase as

$$X_{(\Delta[2] \sqcup_{\Delta[0]} \Delta[2]) \sqcup_{Sp(2)} \Delta[2]}$$

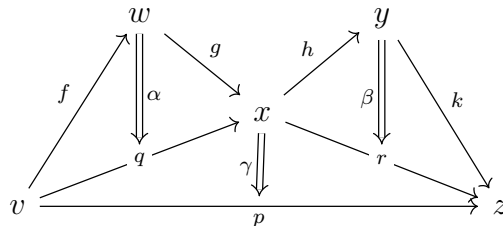
to obtain a lifting problem

$$\begin{array}{ccc}
& & X_{(\Delta[2] \sqcup_{\Delta[0]} \Delta[2]) \sqcup_{Sp(2)} \Delta[2]} \xrightarrow{\tau^*} X_1 \\
& \uparrow \mu & \downarrow \iota^* \\
X_{Sp(4)} & \xrightarrow{id} & X_{Sp(4)}
\end{array}$$

where  $\iota^*$  and  $\tau^*$  are induced by the maps  $\iota$  and  $\tau$  in the cospan in  $\mathbf{sSet}$

$$Sp(4) \xleftarrow{\iota} (\Delta[2] \sqcup_{\Delta[0]} \Delta[2]) \sqcup_{Sp(2)} \Delta[2] \xleftarrow{\tau} \Delta[1].$$

The domain of the map  $\mu$  is well-understood to us by this point: it is simply the  $\infty$ -category of chains of length four in  $X$ . The codomain, however, is new. We should interpret it as the  $\infty$ -category of diagrams in  $X$  of the form



where  $\alpha, \beta, \gamma \in (X_{2,0})_0$  are compositions,  $f, g, h, k, q, r, p \in (X_{1,0})_0$  are morphisms and  $v, w, x, y, z \in (X_{0,0})_0$  are objects in our  $(\infty, 2)$ -category.

The morphism  $\mu$  can now be seen as taking a chain  $v \xrightarrow{f} w \xrightarrow{g} x \xrightarrow{h} y \xrightarrow{k} z$  and extending it to a diagram as above, choosing  $\alpha, \beta, \gamma, q, r$  and  $p$ . The shape of this diagram can in fact tell us directly what the composition operation in question is, as  $p$  is a composite of  $q$  with  $r$ , which are in turn composites of  $f$  with  $g$  and  $h$  with  $k$  respectively.

The general shape of such ‘composition diagrams’ is evident by an induction. In the below definition, write

$$s, t : \Delta[0] \rightarrow Sp(n)$$

for the maps in  $\mathbf{sSet}$  identifying the first and last objects.

**Definition 2.43.** A simplicial composition diagram is a cospan  $Sp(n) \xrightarrow{\iota} K \xleftarrow{\tau} \Delta[1]$  defined inductively such that either:

1.  $K = \Delta[n]$ ,  $\iota = g_n : Sp(n) \hookrightarrow \Delta[n]$  and  $\tau = \langle 0, n \rangle : \Delta[1] \hookrightarrow \Delta[n]$  for some  $n \geq 0$ ;
2.  $K = (K_1 \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} K_d) \sqcup_{Sp(d)} \Delta[d]$  for some  $d > 1$ , where  $Sp(k_i) \xrightarrow{\iota_i} K_i \xleftarrow{\tau_i} \Delta[1]$  are simplicial composition diagrams for  $1 \leq i \leq d$  and the pushouts are taken over the diagrams

$$K_1 \xleftarrow{\iota_1 \circ t} \Delta[0] \xrightarrow{\iota_2 \circ s} \cdots \xleftarrow{\iota_{d-1} \circ t} \Delta[0] \xrightarrow{\iota_d \circ s} K_d$$

and

$$(K_1 \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} K_d) \xleftarrow{\tau_1 \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} \tau_d} Sp(d) \hookrightarrow \Delta[d].$$

The maps are then  $\iota$  as in the diagram

$$\begin{array}{ccc} Sp(n) = Sp(k_1 + \cdots + k_d) & \cong & Sp(k_1) \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} Sp(k_d) \\ & \searrow \iota & \downarrow \iota_1 \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} \iota_d \\ & & K_1 \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} K_d \\ & & \downarrow \\ & & K \end{array}$$

$$\text{and } \tau : \Delta[1] \xrightarrow{\langle 0, d \rangle} \Delta[d] \hookrightarrow K.$$

We will say the arity of the composition diagram  $K$  is  $n$ . Its depth is either 1 if  $K = \Delta[n]$  or  $1 + \max_{1 \leq i \leq d} d_i$  otherwise, where  $d_i$  is the depth of  $K_i$ .

Note the inclusion of  $\Delta[0] \xrightarrow{id} \Delta[0] \xleftarrow{X_{(0,0)}} \Delta[1]$ , which amounts to the problem of taking identities. Note also that it is entirely unambiguous to refer to a simplicial composition diagram solely by the space  $K$ ; there is only one possibility for the maps  $\iota$  and  $\tau$ .

A similar notion to these diagrams is to be found in the membranes obtained from polygonal decompositions of convex  $(n+1)$ -gons used to define 2-Segal spaces in [16]. They also strongly resemble the opetopes of [2]. Making these connections precise and exploring their full influence will be future work.

The space of composition diagrams in question will be  $X_K$  for a complete 2-fold Segal space  $X$  and a simplicial composition diagram  $K$ . It is the case that these are always complete Segal spaces:

**Proposition 2.32.** *Let  $X$  be a complete 2-fold Segal space. Suppose  $K$  is a simplicial composition diagram. Then  $X_K$  is a complete Segal space.*

*Proof.* We proceed by induction on the depth of  $K$ . The result is clear if  $K = \Delta[n]$ . Now, suppose the depth is greater than 1, so  $K = (K_1 \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} K_n) \sqcup_{Sp(n)} \Delta[n]$ . This implies that  $X_K$  is isomorphic to the pullback

$$(X_{K_1} \times_{X_0} \cdots \times_{X_0} X_{K_n}) \times_{X_{Sp(n)}} X_n.$$

The outermost pullback is in fact along a Segal map. Hence, the induced map by the pullback

$$X_K \rightarrow X_{K_1} \times_{X_0} \cdots \times_{X_0} X_{K_n}$$

is a trivial Reedy fibration, meaning it is one in *CSSP*. By induction, the codomain is fibrant in this model structure, so the space  $X_K$  is also fibrant as desired.  $\square$

Moreover, the map  $\iota$  always induces a trivial fibration:

**Proposition 2.33.** *Let  $X$  be a complete 2-fold Segal space. Suppose  $Sp(n) \xrightarrow{\iota} K \xleftarrow{\tau} \Delta[1]$  is a simplicial composition diagram. Then the map  $\iota^* : X_K \rightarrow X_{Sp(n)}$  induced by  $\iota$  is a trivial fibration in *CSSP*.*

*Proof.* By induction on the definition of  $K$ , the map  $\iota^*$  is a pullback of Segal maps and identities and is thus a trivial fibration by the Segal condition on  $X$ .  $\square$

Note also that the maps  $\iota^*$  are tautologically object-fibered with respect to the maps induced by  $\Delta[0] \hookrightarrow Sp(n) \xrightarrow{\iota} K$ . The same can be said for  $\tau^* : X_K \rightarrow X_1$  with respect to the maps  $\Delta[0] \hookrightarrow \Delta[1] \xrightarrow{\tau} K$ . These two maps also agree with the two endpoint inclusions of the spine composed with  $\iota$ .

Simplicial composition diagrams can be used to nest existing composition operations in an evident way, a natural extension of our example:

**Definition 2.44.** *Let  $X$  be a complete 2-fold Segal space with a choice of horizontal compositions  $(\mu_n)_{n \geq 0}$ . Let  $Sp(n) \xrightarrow{\iota} K \xleftarrow{\tau} \Delta[1]$  be a simplicial composition diagram. Then the induced composition on  $K$  is the map  $\mu_K$  solving the lifting problem*

$$\begin{array}{ccc} & & X_K \xrightarrow{\tau^*} X_1 \\ & \nearrow \mu_K & \downarrow \iota^* \\ X_{Sp(n)} & \xrightarrow{id} & X_{Sp(n)} \end{array}$$

where  $\iota^*$  and  $\tau^*$  are induced by  $\iota$  and  $\tau$  respectively, such that:

1. If  $K = \Delta[n]$  then  $\mu_K = \mu_n$ ;
2. If  $K = (K_1 \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} K_r) \sqcup_{Sp(r)} \Delta[r]$  where each  $K_i$  has arity  $k_i$ , then  $\mu_K$  is the pullback map induced by  $(\mu_{K_1} \times_{1_{X_0}} \cdots \times_{1_{X_0}} \mu_{K_r})$  and

$$\mu_r \circ (\tau_1^* \times_{1_{X_0}} \cdots \times_{1_{X_0}} \tau_r^*) \circ (\mu_{K_1} \times_{1_{X_0}} \cdots \times_{1_{X_0}} \mu_{K_r})$$

where  $\tau_i^*$  is induced by the map  $\tau_i : \Delta[1] \rightarrow K_i$  for  $1 \leq i \leq r$ .

It is not hard to see how  $\mu_K$ , by induction, defines a nesting of composition operations described in the structure of  $K$ . In particular, if  $K = (K_1 \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} K_r) \sqcup_{Sp(r)} \Delta[r]$ , then the composition operation in question composes the first  $k_1$  morphisms by  $\mu_{K_1}$ , the second  $k_2$  morphisms by  $\mu_{K_2}$  and so on up to  $\mu_{K_r}$ , then composes the resulting chain of  $n$  morphisms by  $\mu_r$ . This can be generalized:

**Proposition 2.34.** *Suppose  $K = (K_1 \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} K_r) \sqcup_{Sp(r)} K_0$  is a simplicial composition diagram of arity  $n$  for diagrams  $\Delta[k_i] \xrightarrow{\iota_i} K_i \xleftarrow{\tau_i} \Delta[1]$ , where  $k_0 = r$ . Then  $\mu_K$  is the pullback map induced by  $(\mu_{K_1} \times_{1_{X_0}} \cdots \times_{1_{X_0}} \mu_{K_r})$  and*

$$Q := \mu_{K_0} \circ (\tau_1^* \times_{1_{X_0}} \cdots \times_{1_{X_0}} \tau_r^*) \circ (\mu_{K_1} \times_{1_{X_0}} \cdots \times_{1_{X_0}} \mu_{K_r}).$$

*Proof.* The proof is by induction on the depth of  $K_0$ . Either  $K_0 = \Delta[n]$ , for which the proof is trivial, or  $K_0 = (K_0^1 \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} K_0^p) \sqcup_{Sp(p)} \Delta[p]$  for simplicial composition diagrams  $\Delta[h_j] \xrightarrow{\iota_j^0} K_0^j \xleftarrow{\tau_j^0} \Delta[1]$ . Suppose we are in the latter case. We have that

$$K = (K'_1 \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} K'_p) \sqcup_{Sp(p)} \Delta[p]$$

for simplicial composition diagrams  $\Delta[q_j] \xrightarrow{\iota'_j} K'_j \xleftarrow{\tau'_j} \Delta[1]$ . We have moreover for each  $1 \leq j \leq p$  that  $K'_j = (H_1^j \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} H_{h_j}^j) \sqcup_{Sp(h_j)} K_0^j$  for simplicial composition diagrams  $\Delta[s_{j,r}] \xrightarrow{\iota_{j,r}} H_r^j \xleftarrow{\tau_{j,r}} \Delta[1]$ . The spaces  $H_r^j$  are clearly just the spaces  $K_i$ .

We wish to show that the induced pullback map

$$\begin{array}{ccc} X_{Sp(n)} & \xrightarrow{\mu_{K_1} \times_{1_{X_0}} \cdots \times_{1_{X_0}} \mu_{K_r}} & X_{K_1 \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} K_r} \\ & \searrow & \downarrow \tau_1^* \times_{1_{X_0}} \cdots \times_{1_{X_0}} \tau_r^* \\ & X_K & \xrightarrow{\quad \Gamma \quad} & X_{K_1 \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} K_r} \\ & \searrow & \downarrow & \downarrow \tau_1^* \times_{1_{X_0}} \cdots \times_{1_{X_0}} \tau_r^* \\ \mu_{K_0} \circ (\tau_1^* \times_{1_{X_0}} \cdots \times_{1_{X_0}} \tau_r^*) \circ (\mu_{K_1} \times_{1_{X_0}} \cdots \times_{1_{X_0}} \mu_{K_r}) & \searrow & X_{K_0} & \xrightarrow{\iota_0^*} & X_{Sp(r)} \end{array}$$

is  $\mu_K$ . Our strategy is to consider the diagram

$$\begin{array}{ccccc} X_{Sp(n)} & & & & \\ & \searrow & & & \\ & X_K & \xrightarrow{\quad \Gamma \quad} & X_{K'_1 \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} K'_p} & \xleftarrow{R} & X_{K_1 \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} K_r} \\ & \downarrow & & \downarrow & & \downarrow \\ Q & X_p & \xrightarrow{\quad \gamma_p \quad} & X_{Sp(p)} & & X_{Sp(r)} \\ & \uparrow & & \uparrow & & \\ & X_{K_0} & \xrightarrow{\quad \iota_0^* \quad} & X_{Sp(p)} & & X_{Sp(r)} \end{array}$$

$((\tau_1^0)^* \times_{1_{X_0}} \cdots \times_{1_{X_0}} (\tau_p^0)^*) \times_{1_{Sp(p)}} 1_{X_p}$

where

$$R := 1_{X_{K_1 \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} K_r}} \times_{1_{X_{Sp(r)}}} (\mu_{K_0^1} \times_{1_{X_0}} \cdots \times_{1_{X_0}} \mu_{K_0^p})$$

and show that the induced morphisms  $X_{Sp(n)} \rightarrow X_{K'_1 \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} K'_p}$  and  $X_{Sp(n)} \rightarrow X_p$  are  $(\mu_{K'_1} \times_{1_{X_0}} \cdots \times_{1_{X_0}} \mu_{K'_p})$  and

$$\mu_p \circ ((\tau_1^0)^* \times_{1_{X_0}} \cdots \times_{1_{X_0}} (\tau_p^0)^*) \circ (\mu_{K'_1} \times_{1_{X_0}} \cdots \times_{1_{X_0}} \mu_{K'_p})$$

respectively. This will imply the induced pullback map is  $\mu_K$  by the inner pullback square.

Note first that, for each  $1 \leq j \leq p$ , by induction on depth we have that the diagram

$$\begin{array}{ccc} X_{Sp(q_j)} & \xrightarrow{\mu_{K'_j}} & X_{K'_j} \\ \mu_{H_1^j} \times_{1_{X_0}} \cdots \times_{1_{X_0}} \mu_{H_1^j} \downarrow & & \\ X_{H_1^j \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} H_{h_j}^j} & \xrightarrow{1_{X_{H_1^j \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} H_{h_j}^j}} \times_{1_{X_{Sp(h_j)}}} \mu_{K_0^j}} & X_{K_0^j} \end{array}$$

commutes. Thus, by taking a pullback of these maps, the first morphism is as required. For  $Q$ , it will suffice to show that the diagram

$$\begin{array}{ccccc} X_{Sp(n)} & & & & \\ \downarrow & \searrow^{\mu_{K_1} \times_{1_{X_0}} \cdots \times_{1_{X_0}} \mu_{K_r}} & & & \\ X_{K'_1 \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} K'_p} & & X_{K_1 \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} K_r} & & \\ \downarrow & & \downarrow^{\tau_1^* \times_{1_{X_0}} \cdots \times_{1_{X_0}} \tau_r^*} & & \\ X_{K_0^1 \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} K_0^p} & \xleftarrow{\mu_{K_0^1} \times_{1_{X_0}} \cdots \times_{1_{X_0}} \mu_{K_0^p}} & X_{Sp(r)} & & \\ \downarrow^{(\tau_0^1)^* \times_{1_{X_0}} \cdots \times_{1_{X_0}} (\tau_0^p)^*} & & \downarrow^{\mu_{K_0}} & & \\ X_{Sp(p)} & & X_{K_0} & & \\ \swarrow^{\mu_p} & & \downarrow & & \\ X_p & \xleftarrow{((\tau_0^1)^* \times_{1_{X_0}} \cdots \times_{1_{X_0}} (\tau_0^p)^*) \times_{1_{Sp(p)}} 1_{X_p}} & X_p & & \end{array}$$

commutes, where the map  $X_{Sp(n)} \rightarrow X_p$  is induced by the other morphisms. The leftmost pentagon commutes by definition, while the bottom right pentagon commutes by the definition of  $\mu_{K_0}$  and the fact that the lowest horizontal map is simply the projection map from the pullback defining  $X_{K_0}$  in terms of  $X_p$  and the spaces  $X_{K_0^j}$ .

For the uppermost pentagon, we consider each  $K'_j$  and respective  $K_0^j$  in turn for varying  $j$ . The respective diagram will commute in each case by induction. Thus, the whole pentagon commutes and the leftmost vertical map is indeed the map we seek.  $\square$

We can package the nested nature of  $\mu_K$  into a genuine statement about composition operations analogous to the maps  $\circ^{x_0, \dots, x_n}$ :

**Notation 2.11.** Let  $X$  be a complete 2-fold Segal space and  $Sp(n) \xrightarrow{\iota} K \xleftarrow{\tau} \Delta[1]$  a simplicial composition diagram. Let  $x_0, \dots, x_n \in (X_{0,0})_0$ . Then define

$$\begin{array}{ccc} \prod_{i=1}^n X(x_{i-1}, x_i) & \hookrightarrow & X_{Sp(n)}^{x_0, x_n} \xrightarrow{\mu_K^{x_0, x_n}} X_K^{x_0, x_n} \\ & \searrow^{\circ_K^{x_0, \dots, x_n}} & \downarrow^{(\tau^*)^{x_0, x_n}} \\ & & X(x_0, x_n) \end{array}$$

Note again that if  $n = 0$ , the domain of  $\circ_K^x$  is the terminal object  $* \cong \{x\}$ .

**Theorem 2.5.** Suppose  $X$  is a complete 2-fold Segal space and  $Sp(k_i) \xrightarrow{\iota_i} K_i \xleftarrow{\tau_i} \Delta[1]$  are simplicial composition diagrams for  $0 \leq i \leq n$ , with  $k_0 = n \geq 1$ . Let  $r = \sum_i k_i$ . Suppose

$$Y = ((x_0^1, \dots, x_{k_1}^1), \dots, (x_0^n, \dots, x_{k_n}^n))$$



is a nested list of elements of  $(X_{0,0})_0$  such that  $x_{k_i}^i = x_0^{i+1}$  for  $i < n$ . Let  $(x_0, \dots, x_r)$  be the flattened version of this list where all  $x_{k_i}^i$  have been removed for  $i < n$ .

Then, setting  $K := (K_1 \sqcup_{\Delta[0]} \dots \sqcup_{\Delta[0]} K_n) \sqcup_{Sp(n)} K_0$ , we have that

$$\circ_K^{x_0, \dots, x_r} = \circ_{K_0}^{x_0^1, x_0^2, \dots, x_0^n, x_{k_n}^n} \circ (\circ_{K_1}^{x_0^1, \dots, x_{k_1}^1} \times \dots \times \circ_{K_n}^{x_0^n, \dots, x_{k_n}^n}).$$

Before we can prove this theorem, we need some notation for compactness:

**Notation 2.12.** For a series of spans  $X_0 \leftarrow A_i \rightarrow X_0$  with  $1 \leq i \leq n$  and  $n > 0$ , write

$$\prod_{X_0}^{1 \leq i \leq n} A_i := A_1 \times_{X_0} \dots \times_{X_0} A_n.$$

If  $n = 0$ , set this to be the terminal object  $*$ .

*Proof.* By Proposition 2.34, it will suffice to prove that the diagram

$$\begin{array}{ccc} \prod_{i=1}^n \prod_{j=1}^{k_i} X(x_{j-1}^i, x_j^i) & \xrightarrow{\cong} & \prod_{i=1}^r X(x_{i-1}, x_i) \\ \downarrow & & \downarrow \\ \prod_{i=1}^n (\prod_{X_0}^{1 \leq i \leq k_i} X_1)^{x_0^i, x_{k_i}^i} & \xrightarrow{\quad} & (\prod_{X_0}^{1 \leq i \leq r} X_1)^{x_0, x_r} \\ \prod_{i=1}^n \mu_{K_i}^{x_0^i, x_{k_i}^i} \downarrow & & \downarrow (\mu_{K_1} \times_{1_{X_0}} \dots \times_{1_{X_0}} \mu_{K_n})^{x_0, x_r} \\ \prod_{i=1}^n X_{K_i}^{x_0^i, x_{k_i}^i} & \xrightarrow{\quad} & X_{K_1 \sqcup_{\Delta[0]} \dots \sqcup_{\Delta[0]} K_n}^{x_0, x_r} \\ \prod_{i=1}^n (\tau_i^*)^{x_0^i, x_{k_i}^i} \downarrow & & \downarrow (\tau_1^* \times_{1_{X_0}} \dots \times_{1_{X_0}} \tau_n^*)^{x_0, x_r} \\ \prod_{i=1}^n X(x_0^i, x_{k_i}^i) & \xrightarrow{\quad} & X_{Sp(n)}^{x_0, x_r} \\ & & \mu_{K_0}^{x_0, x_r} \downarrow \\ & & X_{K_0}^{x_0, x_r} \\ & & (\tau_0^*)^{x_0, x_r} \downarrow \\ & & X(x_0, x_r) \end{array}$$

commutes, where the inclusions are in general of the form  $A \times_{X_0} \{x\} \times_{X_0} B \hookrightarrow A \times_{X_0} B$ , where  $A \rightarrow X_0 \leftarrow B$  is a cospan of simplicial spaces and  $x \in (X_{0,0})_0$ .

One can reduce the problem to checking each of the three squares in the diagram commutes. The first square is immediate, as the two paths are simply a matter of removing objects  $x_i$  in different orders. The second two squares commute trivially; it is a matter of applying the inclusion before or after the vertical morphisms, which has no effect. Hence, the result is immediate, as the two maps being equated are both paths in this diagram.  $\square$

When we come to define homotopy bicategories, our hom-categories will be  $h_1(X(x, y))$  with composition functors  $h_1(\circ^{x_0, \dots, x_n})$ . To obtain coherence isomorphisms like associators, we need an essential technical property of these maps which respect *maps between simplicial composition diagrams*:

**Definition 2.45.** A map of simplicial composition diagrams is a map  $f : K_1 \rightarrow K_2$  in  $\mathbf{sSet}$  for simplicial composition diagrams  $\Delta[n] \xrightarrow{\iota_i} K_i \xleftarrow{\tau_i} \Delta[1]$  for  $i = \{1, 2\}$ , such that  $f \circ \iota_1 = \iota_2$  and  $f \circ \tau_1 = \tau_2$ .

**Definition 2.46.** Let  $\mathbf{SCD}_n$  be the category whose objects are simplicial composition diagrams of arity  $n$  and whose morphisms are maps of simplicial composition diagrams.

**Proposition 2.35.** Let  $Sp(n) \xrightarrow{\iota_i} K_i \xleftarrow{\tau_i} \Delta[1]$  for  $i \in \{1, 2\}$  be two simplicial composition diagrams of arity  $n \geq 0$  with  $f : K_1 \rightarrow K_2$  a map in  $\mathbf{SCD}_n$  between them. Let  $X$  be a complete 2-fold Segal space.

Then for any  $x_0, \dots, x_n \in (X_{0,0})_0$ , there exists a map  $\mu'_{K_2}$  solving the same lifting problem as  $\mu_{K_1}$ , namely giving a commutative diagram

$$\begin{array}{ccc} & & X_{K_1} \\ & \nearrow \mu'_{K_2} & \downarrow \iota_1^* \\ X_{Sp(n)} & \xrightarrow{id} & X_{Sp(n)} \end{array}$$

such that the diagram

$$\begin{array}{ccc} \prod_{i=1}^n X(x_{i-1}, x_i) & \hookrightarrow & X_{Sp(n)}^{x_0, x_n} \xrightarrow{(\mu'_{K_2})^{x_0, x_n}} X_{K_1}^{x_0, x_n} \\ & \searrow \circ_{K_2}^{x_0, \dots, x_n} & \downarrow (\tau_1^*)^{x_0, x_n} \\ & & X(x_0, x_n) \end{array}$$

commutes.

*Proof.* We define  $\mu'_{K_2} : X_{Sp(n)} \xrightarrow{\mu_{K_2}} X_{K_2} \xrightarrow{f^*} X_{K_1}$ . We have that  $\mu'_{K_2}$  is object-fibered, as  $f^*$ 's commutativity with the maps  $\iota_i^*$  implies it is object-fibered. It is then clear that this is a solution to the lifting problem, since  $\iota_1^* \circ \mu'_{K_2} = (f \circ \iota_1)^* \circ \mu_{K_2} = \iota_2^* \circ \mu_{K_2} = id$ . Moreover, the second diagram commutes, since  $\tau_1^* \circ f^* = (f \circ \tau_1)^* = \tau_2^*$  and we obtain the original definition of  $\circ_{K_2}^{x_0, \dots, x_n}$ .  $\square$

The consequence of this result we need is that, for any  $f : K_1 \rightarrow K_2$  as above, the maps  $\circ_{K_i}^{x_0, \dots, x_n}$  are of the form  $\beta \circ \mu^{(i)} \circ \alpha$ , where the two  $\mu^{(i)}$  for  $i \in \{1, 2\}$  are solutions to the same lifting problem. We will find that this implies there is a homotopy between the two  $\circ_{K_i}^{x_0, \dots, x_n}$ , which will descend under  $h_1$  to our associator.

A final object we should define is the following functor:

**Definition 2.47.** Let  $n > 0$  and  $k_1, \dots, k_n \geq 0$ . Define the functor

$$\mathcal{G}_{k_1, \dots, k_n}^n : \mathbf{SCD}_n \times \prod_{i=1}^n \mathbf{SCD}_{k_i} \rightarrow \mathbf{SCD}_{k_1 + \dots + k_n}$$

defined by sending

$$(K, K_1, \dots, K_n) \mapsto (K_1 \sqcup_{\Delta[0]} \dots \sqcup_{\Delta[0]} K_n) \sqcup_{Sp(n)} K$$

with similar behavior for maps.

One might note that this creates a (nonsymmetric) operad  $\mathbf{SCD}_\bullet$  valued in  $\mathbf{Cat}$ . We will not need this fact particularly so we omit a proof, but this same structure will reappear later.

### 3 Bicategories of Varying Biases

Armed with a reasonable understanding of our domain, we are now ready to consider the codomain of our construction. Our goal is to extend the ideas of homotopy categories of complete Segal spaces to complete 2-fold Segal spaces. It would seem sensible then that the result of our construction should be a bicategory of some description.

We believe that a more natural target may be *unbiased bicategories*. Indeed, consider the nature of composition in a complete 2-fold Segal space  $X$ ; given a choice of horizontal compositions  $\mu_n$ , we obtain composition maps  $\circ^{x_0, \dots, x_n}$  for all  $n \geq 2$  rather than merely binary composition. Such operations would be most faithfully represented by an ‘unbiased’ construction, where there are indeed specified operations to compose  $n$  morphisms at once.

In the below definitions and indeed throughout this paper, we take the convention that for natural isomorphisms  $\phi : F \Rightarrow G$ ,  $\psi : G \Rightarrow H$  and  $\theta : Q \Rightarrow R$ , vertical composition is written  $\psi\phi : F \Rightarrow H$  and horizontal is  $\theta \circ \phi : Q \circ F \Rightarrow R \circ G$ .

**Definition 3.1** ([31, Def. 1.2.1]). *An unbiased bicategory  $\mathcal{B}$  is a collection of the following data:*

1. A collection of objects  $\mathbf{ob}(\mathcal{B})$ ;
2. For each pair of objects  $x$  and  $y$ , a category of 1-morphisms  $x \rightarrow y$  and 2-morphisms  $f \Rightarrow g$  between them,  $\mathbf{Hom}_{\mathcal{B}}(x, y)$ ;
3. For each tuple of objects  $X = (x_0, \dots, x_n)$  for  $n \in \mathbb{Z}_{>0}$ , a functor

$$\circ^X : \mathbf{Hom}_{\mathcal{B}}(x_0, x_1) \times \dots \times \mathbf{Hom}_{\mathcal{B}}(x_{n-1}, x_n) \rightarrow \mathbf{Hom}_{\mathcal{B}}(x_0, x_n)$$

which we write as

$$(f_1, \dots, f_n) \mapsto (f_1 \circ \dots \circ f_n) = \bigcirc_{i=1}^n f_i$$

for  $f_i \in \mathbf{Hom}_{\mathcal{B}}(x_{i-1}, x_i)$  all 1-morphisms (or all 2-morphisms);

4. For each object  $x$ , a functor  $\circ^x : \star \rightarrow \mathbf{Hom}_{\mathcal{B}}(x, x)$  from the discrete singleton category, identifying an element we denote as  $()$ ;
5. For each  $n \in \mathbb{N}$  and  $k_1, \dots, k_n \in \mathbb{Z}_{\geq 0}$ , for each sequence of tuples  $X_i = (x_0^i, \dots, x_{k_i}^i)$  such that  $x_{k_i}^i = x_0^{i+1}$  for every  $i$ , setting

$$X = (X_1, \dots, X_n)$$

and

$$Y = (x_0^1, \dots, x_{k_1}^1, x_1^2, \dots, x_{k_2}^2, \dots, x_1^n, \dots, x_{k_n}^n)$$

a natural isomorphism

$$\gamma_X : \circ^{(x_0^1, x_{k_1}^1, x_{k_2}^2, \dots, x_{k_n}^n)} \circ (\circ^{X_1} \times \circ^{X_2} \times \dots \times \circ^{X_n}) \Rightarrow \circ^Y$$

sending  $((f_1^1 \circ \dots \circ f_{k_1}^1) \circ \dots \circ (f_1^n \circ \dots \circ f_{k_n}^n))$  to  $(f_1^1 \circ \dots \circ f_{k_n}^n)$ , written levelwise as  $\gamma((f_1^1, \dots, f_{k_1}^1), \dots, (f_1^n, \dots, f_{k_n}^n)) : \bigcirc_{i=1}^n (\bigcirc_{j=1}^{k_i} f_j^i) \rightarrow \bigcirc_{i=1}^n \bigcirc_{j=1}^{k_i} f_j^i$ ;

6. For each pair of objects  $x$  and  $y$ , a natural isomorphism  $\iota_{x,y} : \mathbf{1}_{\mathbf{Hom}_{\mathcal{B}}(x,y)} \Rightarrow \circ^{x,y}$  sending  $f$  to  $(f)$ ,

such that:

1. (associativity) for any  $n, m_1, \dots, m_n \in \mathbb{Z}_{>0}$ , integers  $k_1^1, \dots, k_{m_n}^n \in \mathbb{Z}_{\geq 0}$  and thrice-nested sequence of objects  $((x_{p,q,r})_{r=0}^{k_p^q})_{q=1}^{m_p})_{p=1}^n$  such that  $x_{k_p^q, q, p} = x_{0, q+1, p}$  for  $q < m_p$  and  $x_{k_p^{m_p}, m_p, p} = x_{0, 1, p+1}$  for  $p < n$ , together with any sequence of 1-morphisms  $f_{p,q,r} \in \mathbf{Hom}_{\mathcal{B}}(x_{p,q,r-1}, x_{p,q,r})$  for  $p, q, r \geq 1$ , the diagram

$$\begin{array}{ccccc}
& & \bigcirc_{p=1}^n (\bigcirc_{q=1}^{m_p} (\bigcirc_{r=1}^{k_r^q} f_{p,q,r})) & & \\
& \swarrow \scriptstyle \bigcirc_{p=1}^n \gamma_{D_p} & & \searrow \scriptstyle \gamma_D & \\
\bigcirc_{p=1}^n (\bigcirc_{q=1}^{m_p} \bigcirc_{r=1}^{k_r^q} f_{p,q,r}) & & & & \bigcirc_{p=1}^n \bigcirc_{q=1}^{m_p} (\bigcirc_{r=1}^{k_r^q} f_{p,q,r}) \\
& \searrow \scriptstyle \gamma_E & & \swarrow \scriptstyle \gamma_F & \\
& & \bigcirc_{p=1}^n \bigcirc_{q=1}^{m_p} \bigcirc_{r=1}^{k_r^q} f_{p,q,r} & & 
\end{array}$$

commutes, where

- (a)  $D_p = ((f_{p,q,r})_{r=1}^{k_q^p})_{q=1}^{m_p}$ ;  
(b)  $D = ((\bigcirc_{r=1}^{k_q^p} f_{p,q,r})_{q=1}^{m_p})_{p=1}^n$ ;  
(c)  $E = ((f_{p,q,r})_{r=1, q=1}^{k_q^p, m_p})_{p=1}^n$ ;  
(d)  $F = ((f_{p,q,r})_{r=1}^{k_q^p})_{q=1, p=1}^{m_q, n}$ ;

2. (unitality) for any sequence of objects  $x_0, \dots, x_n \in \mathbf{ob}(\mathcal{B})$  and 1-morphisms  $f_i \in \mathbf{Hom}_{\mathcal{B}}(x_{i-1}, x_i)$ , the diagram

$$\begin{array}{ccccc}
& & \bigcirc_{i=1}^n f_i & & \\
& \swarrow \scriptstyle \iota_{\bigcirc_{i=1}^n f_i} & \downarrow & \searrow \scriptstyle \bigcirc_{i=1}^n \iota_{f_i} & \\
(\bigcirc_{i=1}^n f_i) & & \mathbf{1}_{\bigcirc_{i=1}^n f_i} & & \bigcirc_{i=1}^n (f_i) \\
& \searrow \scriptstyle \gamma_{((f_1, \dots, f_n))} & \downarrow & \swarrow \scriptstyle \gamma_{((f_1), \dots, (f_n))} & \\
& & \bigcirc_{i=1}^n f_i & & 
\end{array}$$

commutes.

Note the existence of object identities given by the elements  $(\ )$ . The unitality condition on these is now implicit in the associators, rather than the unitors, which now take a role closer to the unit of a monad rather than a monoid.

We must also define pseudofunctors between these unbiased bicategories:

**Definition 3.2** ([31, Def. 1.2.3]). An unbiased pseudofunctor, or unbiased weak functor  $P : \mathcal{B} \rightarrow \mathcal{C}$  between unbiased bicategories  $\mathcal{B}$  and  $\mathcal{C}$  is the following collection of data:

1. A mapping of objects  $\mathbf{ob}(\mathcal{B}) \rightarrow \mathbf{ob}(\mathcal{C})$ , mapping  $x \mapsto P(x)$ ;

2. For each pair of objects  $x$  and  $y$  in  $\mathcal{B}$ , a functor

$$P_{x,y} : \mathbf{Hom}_{\mathcal{B}}(x, y) \rightarrow \mathbf{Hom}_{\mathcal{C}}(P(x), P(y));$$

3. For each  $n \in \mathbb{Z}_{>0}$  and objects  $x_0, \dots, x_n$ , a natural isomorphism

$$\pi_{x_0, \dots, x_n} : \circ_{\mathcal{C}}^{P(x_0), \dots, P(x_n)} \circ (P_{x_0, x_1} \times \dots \times P_{x_{n-1}, x_n}) \Rightarrow P_{x_0, x_n} \circ_{\mathcal{B}}^{x_0, \dots, x_n}$$

which is levelwise a morphism  $\pi_{(f_1, \dots, f_n)} : \bigcirc_{i=1}^n P_{x_{i-1}, x_i}(f_i) \rightarrow P_{x_0, x_n}(\bigcirc_{i=1}^n f_i)$  for  $f_i \in \mathbf{Hom}_{\mathcal{B}}(x_{i-1}, x_i)$ ;

4. A natural isomorphism  $\pi_x : \circ_{\mathcal{C}}^{P(x)} \Rightarrow P_{x,x} \circ \circ_{\mathcal{B}}^x$  of the form  $() \rightarrow P(() )$

such that:

1. For any  $n > 0$ , integers  $k_1, \dots, k_n \in \mathbb{Z}_{\geq 0}$  and nested sequence of objects  $((x_j^i)_{j=0}^{k_i})_{i=1}^n$  in  $\mathcal{B}$  such that  $x_{k_i}^i = x_0^{i+1}$  for  $i < n$ , along with morphisms  $f_j^i \in \mathbf{Hom}_{\mathcal{B}}(x_{j-1}^i, x_j^i)$  for  $1 \leq i \leq n$  and  $1 \leq j \leq k_i$ , the diagram

$$\begin{array}{ccc} \bigcirc_{i=1}^n (\bigcirc_{j=1}^{k_i} P_{x_{j-1}^i, x_j^i}(f_j^i)) & \xrightarrow{\gamma_D} & \bigcirc_{i=1}^n \bigcirc_{j=1}^{k_i} P_{x_{j-1}^i, x_j^i}(f_j^i) \\ \bigcirc_{i=1}^n \pi_{(f_1^i, \dots, f_{k_i}^i)} \downarrow & & \downarrow \pi_{(f_1^1, \dots, f_{k_n}^n)} \\ \bigcirc_{i=1}^n P_{x_0^i, x_{k_i}^i}(\bigcirc_{j=1}^{k_i} f_j^i) & & \\ \pi_{(\bigcirc_{j=1}^{k_1} f_j^1, \dots, \bigcirc_{j=1}^{k_n} f_j^n)} \downarrow & & \\ P_{x_0^1, x_{k_n}^n}(\bigcirc_{i=1}^n (\bigcirc_{j=1}^{k_i} f_j^i)) & \xrightarrow{P_{x_0^1, x_{k_n}^n}(\gamma_{D'})} & P_{x_0^1, x_{k_n}^n}(\bigcirc_{i=1}^n \bigcirc_{j=1}^{k_i} f_j^i) \end{array}$$

commutes, where:

$$(a) D = ((P_{x_{j-1}^i, x_j^i}(f_j^i))_{j=1}^{k_i})_{i=1}^n;$$

$$(b) D' = ((f_j^i)_{j=1}^{k_i})_{i=1}^n;$$

2. For each  $x, y \in \mathbf{ob}(\mathcal{B})$  and  $f \in \mathbf{Hom}_{\mathcal{B}}(x, y)$ , the diagram

$$\begin{array}{ccc} P_{x,y}(f) & \xrightarrow{\iota_{P_{x,y}(f)}} & (P_{x,y}(f)) \\ & \searrow P_{x,y} \iota_a & \downarrow \pi(f) \\ & & P_{x,y}((f)) \end{array}$$

commutes.

Note again the subtle insertion of  $()$  here, with  $\pi_{x_0} : () \rightarrow P_{x_0, x_0}(() )$ .

We wish to show that these form a category **UBicat** of small unbiased bicategories and unbiased pseudofunctors. We are thus obliged to define the composite of two unbiased pseudofunctors:

**Definition 3.3** ([31, pg. 7-8]). Let  $\mathcal{B}, \mathcal{C}$  and  $\mathcal{D}$  be unbiased bicategories. Let  $P : \mathcal{B} \rightarrow \mathcal{C}$  and  $Q : \mathcal{C} \rightarrow \mathcal{D}$  be unbiased pseudofunctors, with natural isomorphisms  $\pi$  and  $\theta$ , respectively. Then define  $Q \circ P : \mathcal{B} \rightarrow \mathcal{D}$  to be the unbiased pseudofunctor with:

1. Objects mapping as  $x \mapsto Q(P(x))$ ;
2.  $(Q \circ P)_{x,y} := Q_{P(x),P(y)} \circ P_{x,y}$  for  $x, y \in \mathbf{ob}(\mathcal{B})$ ;
3. Natural isomorphisms

$$\psi_{x_0, \dots, x_n} := (Q_{P(x_0), P(x_n)}(\pi_{x_0, \dots, x_n}))(\theta_{P(x_0), \dots, P(x_n)} \circ (P_{x_0, x_1} \times \dots \times P_{x_{n-1}, x_n}))$$

which are thus levelwise of the form

$$\bigcirc_{i=1}^n Q_{P(x_{i-1}), P(x_i)}(P_{x_{i-1}, x_i}(f_i)) \rightarrow Q_{P(x_0), P(x_n)}(P_{x_0, x_n}(\bigcirc_{i=1}^n f_i));$$

4. Natural isomorphisms  $\psi_x := (Q_{x,x}(\pi_x))\theta_{P(x)}$ .

That this defines a genuine pseudofunctor is routine. One may also check that this composition is associative and unital, with identity pseudofunctors clearly being identities on objects and hom-categories with trivial natural isomorphisms. Thus, we have the following:

**Proposition 3.1** ([31, pg. 8]). *There is a category  $\mathbf{UBicat}$ , whose objects are unbiased bicategories, morphisms are unbiased pseudofunctors with composition defined as in Definition 3.3.*

We thus might imagine that we seek a functor

$$h_2 : \mathbf{CSSP}_2 \rightarrow \mathbf{UBicat}.$$

However, complications will arise over the domain for the functor  $h_2$ . Indeed, somehow we will need to choose the horizontal composites  $\mu_n$  for every complete 2-fold Segal space such that they result in a genuine functor. We choose to have a domain as follows:

**Definition 3.4.** *Let  $\mathbf{CSSP}_2^{comp}$  be the category whose objects are pairs  $(X, (\mu_n)_{n \geq 0})$  of complete 2-fold Segal spaces with a choice of horizontal composition and morphisms  $(X, (\mu_n)_{n \geq 0}) \rightarrow (Y, (\nu_n)_{n \geq 0})$  are maps  $X \rightarrow Y$  in  $\mathbf{CSSP}_2$ .*

Note how the morphisms do not account for the choice of horizontal compositions in either the domain or codomain.

We hence seek a functor

$$h_2 : \mathbf{CSSP}_2^{comp} \rightarrow \mathbf{UBicat}.$$

### 3.1 Classical Versus Unbiased Bicategories

One might balk at the definition of an unbiased bicategory. Why make use of such a definition when classical bicategories are better understood and more widely used? They have existed since Bénabou's definition in [5] and remain one of the oldest and best-understood instances of a higher category.

Such questions can be answered with the insight that nothing has truly been lost. Unbiased bicategories can be converted into bicategories easily enough:

**Definition 3.5** ([31, pg. 10]). *Let  $\mathcal{B}$  be an unbiased bicategory. The underlying bicategory  $\tilde{\mathcal{B}}$  is the classical bicategory given by the following data:*

1. Objects and hom-categories are as in  $\mathcal{B}$ ;
2. The composition functors  $\circ^{x,y,z} : \mathbf{Hom}_{\tilde{\mathcal{B}}}(x,y) \times \mathbf{Hom}_{\tilde{\mathcal{B}}}(y,z) \rightarrow \mathbf{Hom}_{\tilde{\mathcal{B}}}(x,z)$  for  $x,y,z \in \mathbf{ob}(\tilde{\mathcal{B}})$  are the binary composition functors in  $\mathcal{B}$ ;
3. Identity functors  $1_x : * \rightarrow \mathbf{Hom}_{\tilde{\mathcal{B}}}(x,x)$  for all  $x \in \mathbf{ob}(\tilde{\mathcal{B}})$  are given by the functors  $\circ^x$  in  $\mathcal{B}$ ;
4. The associators  $\gamma_{w,x,y,z}$  for  $w,x,y,z \in \tilde{\mathcal{B}}$  are given by the tuples  $X_1 := ((w,x),(x,y,z))$  and  $X_2 := ((w,x,y),(y,z))$ , along with the composites

$$\alpha_{w,x,y,z} := (1_{\circ^{w,y,z}} \circ (1_{\circ^{x,y,z}} \times l_{y,z}^{-1})) \gamma_{X_2}^{-1} \gamma_{X_1} (1_{\circ^{w,x,z}} \circ (l_{w,x} \times 1_{\circ^{x,y,z}}))$$

of domain  $\circ^{w,x,z} \circ (1_{\mathbf{Hom}_{\tilde{\mathcal{B}}}(w,x)} \times \circ^{x,y,z})$  and codomain  $\circ^{w,y,z} \circ (\circ^{w,x,y} \times 1_{\mathbf{Hom}_{\tilde{\mathcal{B}}}(y,z)})$ ;

5. The unitors  $\lambda_{x,y}$  and  $\rho_{x,y}$  for  $x,y \in \mathbf{ob}(\tilde{\mathcal{B}})$  are given by the natural isomorphisms

$$\lambda_{x,y} := l_{x,y}^{-1} \gamma_{(x),(x,y)} (1_{\circ^{x,x,y}} \circ (1_{\circ^x} \times l_{x,y})) : \circ^{x,x,y} \circ (\circ^x \times 1_{\mathbf{Hom}_{\tilde{\mathcal{B}}}(x,y)}) \Rightarrow 1_{\mathbf{Hom}_{\tilde{\mathcal{B}}}(x,y)}$$

and

$$\rho_{x,y} := l_{x,y}^{-1} \gamma_{(x,y),(y)} (1_{\circ^{x,y,y}} (l_{x,y} \times 1_{\circ^y})) : \circ^{x,y,y} \circ (1_{\mathbf{Hom}_{\tilde{\mathcal{B}}}(x,y)} \times \circ^y) \Rightarrow 1_{\mathbf{Hom}_{\tilde{\mathcal{B}}}(x,y)}.$$

The pentagon and triangle axioms may be checked to hold from the axioms of an unbiased bicategory. Indeed, the pentagon axiom quickly follows from pondering the following diagram:

$$\begin{array}{ccccccc}
 & & & & ((fg)h)k & & \\
 & & & & \swarrow \text{dotted} & & \searrow \text{dotted} \\
 (f(gh))k & \xrightarrow{\text{dotted}} & (fgh)k & \xrightarrow{\text{solid}} & fghk & \xleftarrow{\text{solid}} & (fg)hk & \xrightarrow{\text{dotted}} & (fg)(hk) \\
 & & \swarrow \text{dotted} & & \uparrow \text{solid} & & \swarrow \text{dotted} & & \\
 & & f(gh)k & & f(ghk) & & fg(hk) & & \\
 & & \uparrow \text{dotted} & & \uparrow \text{dotted} & & \uparrow \text{dotted} & & \\
 & & f((gh)k) & \xrightarrow{\text{dotted}} & & \xrightarrow{\text{dotted}} & f(g(hk)) & & 
 \end{array}$$

The dotted arrows are the newly constructed associators in the biased bicategory, while the dashed and solid arrows are associators in the original unbiased bicategory. Commutativity now follows from all the five triangles and quadrilaterals in the pentagon's interior commuting. The triangles commute by definition, while the quadrilaterals are each an instance of the associativity condition of an unbiased bicategory. Thus, the result holds.

One may also construct a pseudofunctor between classical bicategories from an unbiased pseudofunctor, just by restricting to binary composition. The laws of a pseudofunctor are then induced. This conversion induces a functor  $\tilde{\bullet} : \mathbf{UBicat} \rightarrow \mathbf{Bicat}$ , to the category of bicategories and pseudofunctors:

**Theorem 3.1** ([31, Thm. 1.3.1]). *For any bicategory  $\mathcal{B}$ ,  $\tilde{\mathcal{B}}$  is a bicategory. Moreover,  $\tilde{\bullet}$  defines a genuine functor from  $\mathbf{UBicat}$  to  $\mathbf{Bicat}$ . This functor is fully faithful and essentially surjective.*

It is clear however that there is no ‘canonical’ inverse to the functor  $\tilde{\mathbf{b}}$ . There are many ways to obtain an unbiased bicategory from a biased one, with a countably infinite number of choices resulting just from building the unbiased compositions out of different nestings of binary composition operations. We will not explore this further here.

## 4 Homotopy Categories of Complete Segal Spaces

Before we can proceed with homotopy bicategories, there are a few things we need to prove about homotopy categories of complete Segal spaces, since these will form the foundation of our construction. The first is a natural extension of  $\pi_0$  preserving products:

**Lemma 4.1.**  *$h_1$  preserves products up to natural isomorphism of categories.*

*Proof.* Let  $X$  and  $Y$  be complete Segal spaces. We seek to specify an isomorphism in  $\mathbf{Cat}$  of the form  $p : h_1(X \times Y) \rightarrow h_1(X) \times h_1(Y)$ , which is natural in  $X$  and  $Y$ .

On objects, there is an evident natural bijection  $\mathbf{ob}(h_1(X \times Y)) \rightarrow \mathbf{ob}(h_1(X) \times h_1(Y))$ . For morphisms, note that

$$\begin{aligned} (X \times Y)((x_1, y_1), (x_2, y_2)) &:= \{(x_1, y_1)\} \times_{X_0 \times Y_0} (X_1 \times Y_1) \times_{X_0 \times Y_0} \{(x_2, y_2)\} \\ &\cong (\{x_1\} \times_{X_0} X_1 \times_{X_0} \{x_2\}) \times (\{y_1\} \times_{Y_0} Y_1 \times_{Y_0} \{y_2\}) \\ &= X(x_1, x_2) \times Y(x_2, y_2). \end{aligned}$$

Hence, there are natural bijections

$$\mathbf{Hom}_{h_1(X \times Y)}((x_1, y_1), (x_2, y_2)) \rightarrow \mathbf{Hom}_X(x_1, x_2) \times \mathbf{Hom}_Y(y_1, y_2)$$

induced by  $\pi_0$  preserving products. That these respect identities and composition is easily verified.  $\square$

Henceforth, we will wish to work only with complete Segal spaces. Thus, we will write  $h_1^c : \mathbf{CSSP} \rightarrow \mathbf{Cat}$  to denote the restriction of  $h_1$  to this subcategory.

Another important aim for us is to show that homotopy categories respect left homotopies as natural isomorphisms. Our eventual aim is the following result:

**Theorem 4.2.** *The functor  $h_1^c$  is enriched in  $\mathbf{sSet}$ , where the 1-simplices in the enrichment of the domain are left homotopies and are natural isomorphisms in the codomain.*

To understand what this means, we need a suitable interpretation of the categories  $\mathbf{CSSP}$  and  $\mathbf{Cat}$  as  $\mathbf{sSet}$ -enriched categories. The latter is classical, though we will need a quick definition before we can present a version of it suitable for our methods:

**Definition 4.1.** *Let  $I[n]$  be the groupoid whose object set is  $[n]$  and with all hom-sets singletons.*

In particular, one might notice that  $I[n]$  is isomorphic to a full subcategory of the fundamental groupoid  $\Pi_1(\Delta_t[n])$  of the  $n$ -simplex whose objects are the ‘corners’ of  $\Delta_t[n]$ , namely those points  $(x_0, \dots, x_n)$  where some  $x_i = 1$ . This is equivalent to the entirety of  $\Pi_1(\Delta_t[n])$  along the inclusion functor, a fact quickly proven by the  $n$ -simplex being contractible.



**Definition 4.2.** Let  $\mathbf{Cat}_s$  be the  $\mathbf{sSet}$ -enriched category of small categories with hom-spaces of the form

$$\mathbf{Hom}_{\mathbf{Cat}_s}(\mathcal{C}, \mathcal{D}) := \mathbf{nerve}(\mathbf{Iso}(\mathbf{Fun}(\mathcal{C}, \mathcal{D}))).$$

We choose to write the  $n$ -simplices of this simplicial set as

$$\mathbf{Hom}_{\mathbf{Cat}_s}(\mathcal{C}, \mathcal{D})_n := \mathbf{Hom}_{\mathbf{Cat}}(\mathcal{C} \times I[n], \mathcal{D}).$$

There is a clear cosimplicial object  $I[\bullet] : \Delta \rightarrow \mathbf{Grpd}$  sending each  $[n]$  to  $I[n]$ , defining the simplicial maps here. To see how the above description of natural isomorphisms applies, note that a natural isomorphism  $F \cong G$  of functors  $F, G : \mathcal{C} \rightarrow \mathcal{D}$  can be alternatively be written as a functor  $\mathcal{C} \times I[1] \rightarrow \mathcal{D}$ .

## 4.1 Cosimplicial Resolutions

The story for complete Segal spaces demands some more advanced technology. We could certainly state that it is enriched in  $\mathbf{sSet}$  almost immediately, using  $\mathbf{Map}_1^0(-, -)$ . This is however not the structure we seek. Our wish is for the 1-simplices in our enrichment to represent left homotopies, which should be sent by  $h_1^c$  to natural isomorphisms. Let  $X, Y \in \mathbf{CSSP}$ . A 1-simplex in  $\mathbf{Map}_1^0(X, Y)$  is a morphism  $X \square_1^0 \Delta[1] \rightarrow Y$ . If we apply  $h_1^c$  to this, we hope to obtain an isomorphism  $h_1^c(X) \times I[1] \rightarrow h_1^c(Y)$ .

Suppose  $X \cong *$ . Then we should expect that  $h_1^c(\iota_1^0 \Delta[1]) \cong I[1]$ . However, we find

$$\mathbf{Hom}_{h_1^c(\iota_1^0 \Delta[1])}(0, 1) = \pi_0(\{0\} \times_{\Delta[1]} \Delta[1] \times_{\Delta[1]} \{1\}) = \emptyset.$$

Hence, this category cannot possibly be equivalent to  $I[1]$ , as it is disconnected. Indeed, it is equivalent instead to the discrete category with two objects.

A core issue here is that the constant bisimplicial set  $\iota_1^0 \Delta[1]$  is not Reedy fibrant. For instance, the demand that the map  $\mathbf{Map}_1^0(F_1^0(1), \iota_1^0 \Delta[1]) \rightarrow \mathbf{Map}_1^0(\partial F_1^0(1), \iota_1^0 \Delta[1])$  is a fibration in  $\mathbf{sSet}$  is asking for the diagonal map  $(id, id) : \Delta[1] \rightarrow \Delta[1] \times \Delta[1]$  to be a fibration in  $\mathbf{sSet}$ . This is clearly not the case, as we can set up a horn lifting problem

$$\begin{array}{ccc} \Delta[0] & \xleftarrow{\langle 0 \rangle} & \Delta[1] \\ \langle 0 \rangle \downarrow & & \downarrow (id, id) \\ \Delta[1] & \xrightarrow{(id, \langle 0, 0 \rangle)} & \Delta[1] \times \Delta[1] \end{array}$$

which has no solution. Moreover, the simplicial set  $\Delta[1]$  is itself not a Kan complex - the map

$$\langle 0, 1 \rangle \sqcup_{\langle 0 \rangle} \langle 0, 0 \rangle : \Delta[1] \sqcup_{\langle 0 \rangle, \Delta[0], \langle 0 \rangle} \Delta[1] \cong \Lambda_0^2 \rightarrow \Delta[1]$$

has no filler to  $\Delta[2] \rightarrow \Delta[1]$ , so the level 0 condition for Reedy fibrancy also fails. We might interpret this failure as the fact that the poset category of two objects and one morphism between them is not an  $\infty$ -groupoid, as this one morphism has no inverse. This observation also implies that  $\Delta[1]$  is not projective fibrant, so cannot possibly be a complete Segal space in the sense of [27] either.

A sensible path forwards is to identify some deeper mechanism which gives rise to the appearance of  $\Delta[1]$  in our theory and consider whether  $\Delta[1]$  can be thus substituted for something more well-behaved, while continuing to reap all the benefits said mechanism

provided. The correct such structure is that of a *homotopy function complex* and the underlying *cosimplicial resolution*. The reason to think about these is that cosimplicial resolutions are closely tied to notions of left homotopy, always exhibiting valid cylinder objects:

**Definition 4.3** ([15, Def. 4.2]). *Let  $\mathcal{M}$  be a model category and  $C$  an object in  $\mathcal{M}$ . A cylinder object of  $C$  is a factorization*

$$C \sqcup C \xrightarrow{i} C \wedge I \xrightarrow{p} C$$

*of the map  $C \sqcup C \rightarrow C$  such that  $p$  is a weak equivalence. The cylinder object is a good cylinder object if  $i$  is a cofibration and a very good cylinder object if it is good and  $p$  is a trivial fibration.*

**Proposition 4.1** ([24, Prop. 16.1.6]). *Let  $\mathcal{M}$  be a model category. If  $\tilde{X}$  is a cosimplicial resolution of  $X$  in  $\mathcal{M}$ , then*

$$\tilde{X}_0 \sqcup \tilde{X}_0 \xrightarrow{\tilde{X}_{(0)} \sqcup \tilde{X}_{(1)}} \tilde{X}_1 \xrightarrow{\tilde{X}_{(0,0)}} \tilde{X}_0$$

*is a good cylinder object of  $\tilde{X}_0$ .*

Recall from Corollary 2.2.3 that  $\mathbf{Map}_1^0(-, -)$  is a left homotopy function complex induced by a cosimplicial resolution whose cylinder objects are of the form  $X \sqcup X \rightarrow X \times \Delta[1] \rightarrow X$ . It seems reasonable then that replacing this cosimplicial resolution is the correct way to replace  $\Delta[1]$ . We should choose the resolution so that the resulting cylinder objects are of the form  $X \sqcup X \rightarrow X \times K \rightarrow X$  for some  $K$  such that  $h_1(K) \cong I[1]$ . This will imply we obtain an enrichment in homotopy function complexes whose 1-simplices are sent by  $h_1$  to natural isomorphisms. We will find that this indeed leads to the enrichment we seek.

Another advantage to this approach to building an enrichment is that our mapping spaces will always be Kan complexes:

**Proposition 4.2** ([24, Prop. 17.1.3]). *Suppose  $\mathcal{M}$  is a model category and  $X$  and  $Y$  are objects of  $\mathcal{M}$ . Then a left homotopy function complex is a Kan complex.*

This will later let us compose our left homotopies vertically.

Our chosen cosimplicial resolution involves the *classifying diagram* functor:

**Definition 4.4** ([42, pg. 8]). *The classifying diagram functor  $N : \mathbf{Cat} \rightarrow \mathbf{sSpace}$  is defined such that for any category  $\mathcal{C}$  and  $n \geq 0$ ,*

$$N\mathcal{C}_n := \mathbf{nerve}(\mathbf{Iso}(\mathcal{C}^{[n]}))$$

*with the evident simplicial maps and behavior on functors. Here,  $\mathbf{Iso} : \mathbf{Cat} \rightarrow \mathbf{Grpd}$  sends a category to its maximal subgroupoid.*

**Proposition 4.3** ([42, Prop. 6.1]). *Let  $\mathcal{C} \in \mathbf{Cat}$ . Then  $N\mathcal{C}$  is a complete Segal space.*

**Proposition 4.4** ([42, pg. 13]).  *$h_1 \circ N = 1_{\mathbf{Cat}}$ .*

*Proof.* The objects of  $h_1(N\mathcal{C})$  are just  $(N\mathcal{C}_0)_0$ , which are the 0-simplices of  $\mathbf{nerve}(\mathbf{Iso}(\mathcal{C}))$ , namely the objects. Hence, the object sets are the same. For morphisms, we note both  $\mathbf{nerve} : \mathbf{Cat} \rightarrow \mathbf{sSet}$  and  $\mathbf{Iso} : \mathbf{Cat} \rightarrow \mathbf{Grpd}$  are right adjoints. Thus, they commute with pullbacks, meaning for any objects  $x$  and  $y$  of  $\mathcal{C}$ ,

$$\mathbf{nerve}(\mathbf{Iso}(\mathcal{C}^{[1]})) \times_{(\mathbf{nerve}(\mathbf{Iso}(\mathcal{C})))^2} \{(x, y)\} \cong \mathbf{nerve}(\mathbf{Iso}(\mathcal{C}^{[1]} \times_{\mathcal{C}^2} \{(x, y)\}))$$

Note that  $\mathcal{C}^{[1]} \times_{\mathcal{C}^2} \{(x, y)\}$  is always discrete as a category - indeed, all morphisms are squares whose objects are  $x$  and  $y$  and whose vertical maps are identities. Hence, the entire nerve is discrete, so the hom-sets of  $h_1(N\mathcal{C})$  are path components of discrete spaces, and so are just the original hom-sets as needed.

Identities are also clearly the same, as is composition. Hence,  $h_1(N\mathcal{C}) \cong \mathcal{C}$  as needed. Functors share a similar fate by inspection, so the proof is complete.  $\square$

For a complete Segal space  $X$ , we take our cosimplicial resolution to be  $\tilde{X} := X \times (N \circ I[\bullet])$ , so that  $\tilde{X}_n := X \times N(I[n])$ . This is still cofibrant by Corollary 2.1.2. Hence, since complete Segal spaces are fibrant in the model structure  $CSSP$ , we can build a Kan complex  $\mathbf{Map}_{\mathbf{sS}}^I(X, Y)$  for any complete Segal spaces  $X$  and  $Y$ , which at level  $k$  is the set of maps  $X \times N(I[k]) \rightarrow Y$ , given by the left homotopy function complex induced by our cosimplicial resolution.

**Proposition 4.5.**  *$\mathbf{Map}_{\mathbf{sS}}^I(-, -) : CSSP^{op} \times CSSP \rightarrow \mathbf{sSet}$  defines an enrichment of  $CSSP$  in Kan complexes.*

*Proof.* The composition operation in question acts via the Cartesian closedness. Suppose  $X, Y$  and  $Z$  are complete Segal spaces and let  $k \geq 0$ . Let  $f : X \times N(I[k]) \rightarrow Y$  and  $g : Y \times N(I[k]) \rightarrow Z$ . Then  $g$  is equivalently a map  $Y \rightarrow Z^{N(I[k])}$ , giving us a composite  $gf : X \times N(I[k]) \rightarrow Z^{N(I[k])}$ , which is equivalently a map  $X \times N(I[k]) \times N(I[k]) \rightarrow Z$ . We have a diagonal map  $I[k] \hookrightarrow I[k] \times I[k]$  sending  $n \mapsto (n, n)$  for  $n \in [k]$ . Since  $N$  preserves products [42, Thm. 3.7], this extends to a natural diagonal map  $N(I[k]) \rightarrow N(I[k]) \times N(I[k])$ . Precomposition with this gives the map  $X \times N(I[k]) \rightarrow Z$  as desired.

These operations are compatible with the simplicial structure, giving an operation

$$\circ : \mathbf{Map}_{\mathbf{sS}}^I(X, Y) \times \mathbf{Map}_{\mathbf{sS}}^I(Y, Z) \rightarrow \mathbf{Map}_{\mathbf{sS}}^I(X, Z).$$

Identities are given by  $1_X : X \times N(I[0]) \rightarrow X$ . Associativity and identity laws are routine.  $\square$

We thus obtain our  $\mathbf{sSet}$ -enrichment. Moreover, the enrichment of  $h_1^c$  is now immediate: there is an evident simplicial map for complete Segal spaces  $X, Y$  of the form

$$\mathbf{Map}_{\mathbf{sS}}^I(X, Y) \rightarrow \mathbf{nerve}(\mathbf{Iso}(\mathbf{Fun}(h_1^c(X), h_1^c(Y))))$$

sending  $X \times N(I[k]) \rightarrow Y$  to  $h_1^c(X) \times I[k] \cong h_1^c(X) \times h_1^c(N(I[k])) \cong h_1^c(X \times N(I[k])) \rightarrow h_1^c(Y)$ .

**Proposition 4.6.** *By the above mapping,  $h_1^c$  extends to an  $\mathbf{sSet}$ -enriched functor.*

*Proof.* It is clear that identities are preserved, so we are left to show the same for composition. However, one notes that if a natural isomorphism  $\alpha : F \Rightarrow G$  between functors  $F, G : \mathcal{C} \rightarrow \mathcal{D}$  is written in the form  $\mathcal{C} \times I[1] \rightarrow \mathcal{D}$ , then horizontal composition is exactly as stated for  $\mathbf{Map}_{\mathbf{sS}}^I$  at all levels of  $\mathbf{nerve}(\mathbf{Iso}(\mathbf{Fun}(\mathcal{C}, \mathcal{D})))$ . Hence, composition is preserved as needed.  $\square$

We now have the desired simplicial set enrichment. We also have that the 1-simplices in  $\mathbf{Map}_{\mathbf{SS}}^I(X, Y)$  are valid left homotopies, as  $X \times N(I[1])$  is a cylinder object for  $X$  owing to it being part of a cosimplicial resolution. We thus have established how  $h_1^c$  can transmit left homotopies to natural isomorphisms in an ‘ $\infty$ -groupoidal’ manner.

A useful consequence of this particular choice of cylinder is that it is in fact a very good cylinder object:

**Proposition 4.7.** *Let  $X$  be a simplicial space. Then  $X \times N(I[1])$  is a very good cylinder object for  $X$  in CSSP.*

*Proof.* The map  $X \sqcup X \hookrightarrow X \times N(I[1])$  is clearly levelwise an inclusion, so is levelwise a Reedy cofibration and thus a cofibration as needed. It then suffices to show that the map  $N(I[1]) \rightarrow *$  is a fibration, which is true since classifying diagrams are always complete Segal spaces [42, Prop. 6.1].  $\square$

One final fact we will need about left homotopies is their capacity to be composed. We need not prove this in full generality:

**Definition 4.5.** *Consider a lifting problem of complete Segal spaces*

$$\begin{array}{ccc} & & B \\ & & \downarrow p \\ C & \longrightarrow & D \end{array}$$

where  $p$  is a trivial fibration. Suppose  $f, g, h : C \rightarrow B$  are solutions, with left homotopies  $H, K : C \times N(I[1]) \rightarrow B$  over  $D$ , from  $f$  to  $g$  and from  $g$  to  $h$  respectively. Then a composite of  $H$  and  $K$  is a left homotopy  $KH : C \times N(I[1]) \rightarrow B$  over  $D$  from  $f$  to  $h$  defined by the composition  $Q \circ i$  in the diagram

$$\begin{array}{ccccc} & & C \sqcup C \sqcup C & & \\ & & \downarrow & \searrow^{f \sqcup g \sqcup h} & \\ & & C \times (N(I[1]) \sqcup_* N(I[1])) & \overset{H \sqcup_g K}{\dashrightarrow} & B \\ & & \downarrow j & \swarrow^Q & \downarrow p \\ C \times N(I[1]) & \xleftarrow{i} & C \times N(I[2]) & \longrightarrow & D \end{array}$$

where  $i$  is the image of  $\langle 0, 2 \rangle : \Delta[1] \rightarrow \Delta[2]$  in the cosimplicial object  $C \times N(I[\bullet])$ , while  $j$  on each component  $C \times N(I[1])$  in its domain is the image of the maps  $\langle 0, 1 \rangle$  and  $\langle 1, 2 \rangle$  respectively.

**Proposition 4.8.** *In the situation of Definition 4.5, the map  $Q$  is a horn filler for the 2-horn  $\Lambda_1^2 \rightarrow \mathbf{Map}_{\mathbf{SS}}^I(C, B)$  defined by  $H$  and  $K$  on each of its respective 1-simplices. The homotopy  $KH$  represents the edge  $\langle 0, 2 \rangle$  of  $Q$ .*

Hence, it is reasonable to see  $KH$  as a valid composite of  $H$  and  $K$  in the Kan complex  $\mathbf{Map}_{\mathbf{SS}}^I(C, B)$ . Of course,  $KH$  is not at all unique. This will not matter when we come to need this result.

## 4.2 Globular 2-Homotopies

We are in need of some result that establishes when  $h_1^c$  sends two left homotopies  $K, H : X \times N(I[1]) \rightarrow Y$  between the same maps  $f, g : X \rightarrow Y$  to the same natural isomorphism. A suitable such criterion is the existence of a *globular 2-homotopy* between  $K$  and  $H$ :

**Definition 4.6.** *Let  $X, Y, f, g, K, H$  be as above. Then a globular 2-homotopy  $\alpha : K \Rightarrow H$  is a left homotopy from  $K$  to  $H$ , namely a functor  $X \times N(I[1]) \times N(I[1]) \rightarrow Y$ , such that the restrictions to  $X \times \{0\} \times N(I[1])$  and  $X \times \{1\} \times N(I[1])$  are constantly  $f$  and  $g$  respectively.*

**Proposition 4.9.** *Let  $X, Y, f, g, K, H$  be as above. Suppose  $\alpha : K \Rightarrow H$  is a globular 2-homotopy from  $K$  to  $H$ . Then  $h_1^c(K) = h_1^c(H) : h_1^c(X) \times I[1] \rightarrow Y$ .*

*Proof.* This is a matter of unwinding definitions. We find that

$$h_1^c(\alpha) : h_1^c(X) \times I[1] \times I[1] \rightarrow Y$$

is a map that reduces to  $h_1^c(f)$  on  $h_1^c(X) \times \{0\} \times I[1]$  and  $h_1^c(g)$  on  $h_1^c(X) \times \{1\} \times I[1]$ . Moreover, it restricts to  $h_1^c(K)$  and  $h_1^c(H)$  on the other evident restrictions. It is then clear that  $h_1^c(K) = h_1^c(H)$  as needed.  $\square$

An important consequence to this fact for us lies in comparing solutions of lifting problems. Consider a lifting problem with two possible solutions

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow i & \nearrow \delta & \downarrow p \\ C & \xrightarrow{g} & D \\ & \nwarrow \epsilon & \end{array}$$

where  $i$  is a cofibration and  $p$  a trivial fibration. We have the following useful result:

**Proposition 4.10** ([24, Prop. 7.6.13]). *Let  $\mathcal{M}$  be a model category. Consider the above solid arrow diagram, where adding in either  $\delta$  or  $\epsilon$  alone still makes the diagram commute. Then there is a left homotopy from  $\delta$  to  $\epsilon$  in the model category  $(A \downarrow \mathcal{M} \downarrow D)$ .*

It will be important for us to understand how this is proven.

*Proof.* There is a model structure on  $(A \downarrow \mathcal{M} \downarrow D)$  given in [24, Thm. 7.6.5], where weak equivalences, fibrations and cofibrations are those which restrict to such maps in  $\mathcal{M}$ .

Now, factor the map  $C \sqcup_A C \rightarrow C$  as  $C \sqcup_A C \xrightarrow{j} C \wedge I \xrightarrow{r} C$  as a good cylinder object, so  $j$  is a cofibration and  $r$  a weak equivalence. Consider the solid arrow diagram

$$\begin{array}{ccc} C \sqcup_A C & \xrightarrow{\delta \sqcup_f \epsilon} & B \\ \downarrow j & \nearrow H & \downarrow p \\ C \wedge I & \xrightarrow{r} C \xrightarrow{g} & D \end{array}$$

where  $j$  is a cofibration and  $p$  is a trivial fibration. Thus, there is a dotted arrow  $H$  making the diagram commute, which makes  $H$  a left homotopy from  $\delta$  to  $\epsilon$  in  $(A \downarrow \mathcal{M} \downarrow D)$  as needed.  $\square$

A side result should be noted with regards to these homotopies and taking fibers:

**Proposition 4.11.** *Suppose  $\mathcal{M} = \text{CSSP}$  in the above lifting problem,  $X$  is a complete 2-fold Segal space and that  $f, i, p, g, \delta$  and  $\epsilon$  are all object-fibered over  $(X_0)^{n+1}$ . Then the induced homotopy  $H$  will also be object-fibered, with respect to the vertex maps  $C \wedge I \rightarrow C \rightarrow (X_0)^{n+1}$ . Moreover, for any  $x_0, \dots, x_n \in (X_{0,0})_0$ , the map  $H^{x_0, \dots, x_n} : (C \wedge I)^{x_0, \dots, x_n} \rightarrow B^{x_0, \dots, x_n}$  is a left homotopy from  $\delta^{x_0, \dots, x_n}$  to  $\epsilon^{x_0, \dots, x_n}$ .*

*Proof.* The map  $j$  is clearly object-fibered, by the naturally induced map from the pushout  $C \sqcup_A C \rightarrow (X_0)^{n+1}$ . The map  $\delta \sqcup_f \epsilon$  must also be object-fibered, while  $p$  and  $g$  are by assumption. Thus, by Proposition 2.35, the solution  $H$  of the lifting problem must also be object-fibered such that commutativity of the diagram is preserved after taking a fiber.

Note that  $r^{x_0, \dots, x_n}$  is a trivial fibration and thus a weak equivalence, so  $(C \wedge I)^{x_0, \dots, x_n}$  is a cylinder object, though perhaps not a good or very good one anymore. Hence,  $H^{x_0, \dots, x_n}$  is a valid left homotopy as claimed.  $\square$

Note that in the case  $C \wedge I = C \times N(I[1])$ , the homotopy will in fact stay a very good left homotopy.

If  $A = \emptyset$  is the initial object, the cylinder object in question is also a cylinder object in  $(\mathcal{M} \downarrow D)$ , as well as in  $\mathcal{M}$ , yielding a lifting problem of the form

$$\begin{array}{ccc} C \sqcup C & \xrightarrow{\delta \sqcup \epsilon} & B \\ \downarrow i & \nearrow H & \downarrow p \\ C \wedge I & \xrightarrow{g} & D \end{array}$$

which we encapsulate in the following corollary:

**Corollary 4.2.1.** *Given a model category  $\mathcal{M}$  and a diagram*

$$\begin{array}{ccc} & & B \\ & \nearrow \delta & \downarrow p \\ C & \xrightarrow{g} & D \end{array}$$

where  $C$  is cofibrant,  $p : B \rightarrow D$  is a trivial fibration and  $\delta$  and  $\epsilon$  are each solutions to the lifting problem, then there is a left homotopy in  $(\mathcal{M} \downarrow D)$  from  $\delta$  to  $\epsilon$ .

We can then compare two such left homotopies  $H, K : C \wedge I \rightarrow B$ . As per the proof of Proposition 4.10, these are two solutions to a new lifting problem, letting us inductively apply the proposition to obtain a left homotopy  $\Gamma$  from  $H$  to  $K$  in  $(C \sqcup C \downarrow \mathcal{M} \downarrow D)$ .

We should spend some time understanding precisely what such a left homotopy is in our use case. Consider, for  $f : X \rightarrow Z$  a map of simplicial spaces and  $p : Y \rightarrow Z$  a trivial fibration in  $\mathbf{sSpace}$  with the model structure  $\text{CSSP}$ , two solutions  $\alpha$  and  $\beta$  of the lifting problem of the form

$$\begin{array}{ccc} & & Y \\ & \nearrow \alpha & \downarrow p \\ X & \xrightarrow{f} & Z \end{array}$$

Then consider two left homotopies  $H, K : \alpha \Rightarrow \beta$  induced as solutions to the lifting problem

$$\begin{array}{ccc}
X \sqcup X & \xrightarrow{\alpha \sqcup \beta} & Y \\
\downarrow & \nearrow H & \downarrow p \\
& & Z \\
X \times N(I[1]) & \xrightarrow{f \circ (1_X \times !)} & Z
\end{array}$$

Then there is an induced left homotopy  $\Gamma$  in  $(X \sqcup X \downarrow \mathbf{sSpace} \downarrow Z)$  from  $H$  to  $K$ . We need a suitable cylinder object in this model category to be able to write down  $\Gamma$  directly. One may produce a general such cylinder object. Before we do so, a technical lemma is needed for the construction:

**Lemma 4.3.** *Suppose  $C \sqcup C \xrightarrow{i} C \wedge I \xrightarrow{p} C$  is a very good cylinder object for  $C$  in a model category  $\mathcal{M}$ . Then there is a very good cylinder object*

$$(C \wedge I) \sqcup (C \wedge I) \xrightarrow{h} C \wedge I \wedge I \xrightarrow{r} C \wedge I$$

for  $C \wedge I$  and a cofibration  $v$  that solves the lifting problem

$$\begin{array}{ccccc}
(C \sqcup C) \sqcup (C \sqcup C) & \xrightarrow{i \sqcup i} & (C \wedge I) \sqcup (C \wedge I) & \xrightarrow{h} & C \wedge I \wedge I \\
\tau \downarrow & & \nearrow v & & \downarrow r \\
(C \sqcup C) \sqcup (C \sqcup C) & & & & \\
i \sqcup i \downarrow & & & & \\
(C \wedge I) \sqcup (C \wedge I) & \xrightarrow{p \sqcup p} & C \sqcup C & \xrightarrow{i} & C \wedge I
\end{array}$$

where  $\tau : (C \sqcup C) \sqcup (C \sqcup C) \rightarrow (C \sqcup C) \sqcup (C \sqcup C)$  is the isomorphism permuting the  $C$ 's by the permutation  $(1324)$ <sup>7</sup>.

*Proof.* Consider the pushout diagram

$$\begin{array}{ccc}
(C \sqcup C) \sqcup (C \sqcup C) & \xrightarrow{i \sqcup i} & (C \wedge I) \sqcup (C \wedge I) \\
\tau \downarrow & & \downarrow h' \\
(C \sqcup C) \sqcup (C \sqcup C) & & \\
i \sqcup i \downarrow & & \lrcorner \\
(C \wedge I) \sqcup (C \wedge I) & \xrightarrow{v'} & C \wedge \square
\end{array}$$

We note that  $h'$  and  $v'$  are both cofibrations, by preservation along pushouts. Now, a

<sup>7</sup>Read this as the map  $(a_1 a_2 a_3 a_4)$  sending  $i \mapsto a_i$ , rather than the cyclic permutation.

universal map  $f : C \wedge \square \rightarrow C \wedge I$  is obtained from the pushout by the diagram

$$\begin{array}{ccc}
(C \sqcup C) \sqcup (C \sqcup C) & \xrightarrow{i \sqcup i} & (C \wedge I) \sqcup (C \wedge I) \\
\tau \downarrow & & \downarrow h' \\
(C \sqcup C) \sqcup (C \sqcup C) & & \\
i \sqcup i \downarrow & \lrcorner & \downarrow \\
(C \wedge I) \sqcup (C \wedge I) & \xrightarrow{v'} & C \wedge \square \\
p \sqcup p \downarrow & & \downarrow f \\
C \sqcup C & \xrightarrow{i} & C \wedge I
\end{array}$$

Factorize this into a cofibration followed by a trivial fibration  $C \wedge \square \xrightarrow{h'} C \wedge I \wedge I \xrightarrow{r} C \wedge I$ . The map  $h$  is then defined to be  $\iota \circ h'$ , concluding the construction of the very good cylinder object. The map  $v$  is then set to be  $\iota \circ v'$ . This is clearly a cofibration and solves the required lifting problem by commutativity of the pushout diagram.  $\square$

Note that for us, given  $\mathcal{M} = \mathbf{sSpace}$ , some complete Segal space  $X$  and using the cylinder object  $X \times N(I[1])$ , Lemma 4.3's induced cylinder object can be set to be  $X \times N(I[1]) \times N(I[1])$ , while  $v$  is given by the evident inclusion of the 'vertical' isomorphisms.

**Lemma 4.4.** *Suppose  $C \sqcup C \xrightarrow{i} C \wedge I \xrightarrow{p} C$  is a very good cylinder object for  $C$  in a left proper model category  $\mathcal{M}$ . Let  $h, r, C \wedge I \wedge I$  and  $v$  be as in Lemma 4.3.*

*Then there is a cylinder object for  $C \wedge I$  in  $(C \sqcup C \downarrow \mathcal{M})$  of the form*

$$(C \wedge I) \sqcup_{C \sqcup C} (C \wedge I) \xrightarrow{b} C \wedge B \xrightarrow{z} C \wedge I$$

where  $C \wedge B$  is the pushout

$$\begin{array}{ccc}
(C \wedge I) \sqcup (C \wedge I) & \xrightarrow{p \sqcup p} & C \sqcup C \\
v \downarrow & & \downarrow s \\
C \wedge I \wedge I & \xrightarrow{k} & C \wedge B.
\end{array}$$

*Proof.* We need to define the two maps  $b$  and  $z$ . It will then be sufficient to show that  $z$  is a weak equivalence and that  $zb$  is the appropriate projection map

$$(C \wedge I) \sqcup_{C \sqcup C} (C \wedge I) \rightarrow C \wedge I$$

induced by the pushout.

The map  $z$  is easiest, so we start here: it is simply given by the universal map from the pushout, induced by the maps  $i : C \sqcup C \rightarrow C \wedge I$  and  $r : C \wedge I \wedge I \rightarrow C \wedge I$ . That these commute is due to the lifting problem  $v$  is a solution for. Note then that  $k$  is a weak equivalence, as  $\mathcal{M}$  is left proper,  $v$  is a cofibration and  $p \sqcup p$  is a weak equivalence. This implies by 2-out-of-3 with  $r$  that  $z$  is a weak equivalence, as needed.

Now we turn to  $b$ . Consider the two morphisms  $i_1, i_2 : C \sqcup C \rightarrow (C \wedge I) \sqcup (C \wedge I)$ . It will suffice to show that  $kh_1 = kh_2$ , as this will induce a pushout map to  $C \wedge B$ .



Extending the pushout diagram defining  $C \wedge B$  reveals two larger commutative diagrams, for  $j \in \{1, 2\}$ , of the form

$$\begin{array}{ccccc}
C \sqcup C & \xrightarrow{\alpha_j} & (C \wedge I) \sqcup (C \wedge I) & \xrightarrow{p \sqcup p} & C \sqcup C \\
i_j \downarrow & & v \downarrow & & \downarrow s \\
(C \wedge I) \sqcup (C \wedge I) & \xrightarrow{h} & C \wedge I \wedge I & \xrightarrow{k} & C \wedge B
\end{array}$$

where  $\alpha_1$  and  $\alpha_2$  are induced by the lifting problem defining  $v$ . Commutativity is given by this same lifting problem. It thus suffices to show that  $s(p \sqcup p)\alpha_1 = s(p \sqcup p)\alpha_2$ . However,  $(p \sqcup p)\alpha_1$  and  $(p \sqcup p)\alpha_2$  give the identity on  $C \sqcup C$ , so they are equal as needed, in turn defining  $b$ .

We finally need to show that  $b$  and  $z$  satisfy the requirements of a cylinder object, namely that  $z \circ b$  is the projection map from the pushout. This is a matter of a quick diagram chase and an application of the same property of  $h$  and  $r$ . Indeed, if  $q : (C \wedge I) \sqcup (C \wedge I) \rightarrow (C \wedge I) \sqcup_{C \sqcup C} (C \wedge I)$  is the natural map, then  $z b q = z k h = r h$  by inspection, which is the required projection.  $\square$

Note that we did not prove this to be a very good cylinder object, as this is not needed for our purposes. In fact, we will never need to work with this strange cylinder object directly:

**Lemma 4.5.** *Suppose, in the situation of Lemma 4.4, we have  $D \in \mathcal{M}$  and two homotopies  $H, K : C \wedge I \rightarrow D$ , both between maps  $f, g : C \rightarrow D$ .*

*Then there is a left homotopy from  $H$  to  $K$  in  $(C \sqcup C \downarrow \mathcal{M})$  if and only if there is a left homotopy*

$$\Gamma : C \wedge I \wedge I \rightarrow D$$

*from  $H$  to  $K$  in  $\mathcal{M}$  such that  $\Gamma \circ v = (f \circ p) + (g \circ p)$ .*

*Proof.* Suppose the map  $\Gamma$  exists. The property it satisfies corresponds exactly to having a map to  $D$  from the span which  $C \wedge B$  is the pushout of. Thus, a left homotopy  $\Gamma' : C \wedge B \rightarrow D$  is induced by universality of the pushout.

Now, suppose  $\kappa : C \wedge B' \rightarrow D$  is a left homotopy in  $(C \sqcup C \downarrow \mathcal{M})$ , for some good cylinder object

$$(C \wedge I) \sqcup_{C \sqcup C} (C \wedge I) \xrightarrow{b'} C \wedge B' \xrightarrow{z'} C \wedge I.$$

Some work is needed to obtain a left homotopy using the cylinder object  $C \wedge B$  itself, as we have not proven it to be a good cylinder object. For now, take any factorization of  $b$

$$(C \wedge I) \sqcup_{C \sqcup C} (C \wedge I) \xrightarrow{\gamma} C \wedge \beta \xrightarrow{\omega} C \wedge B$$

into a cofibration  $\gamma$  followed by a trivial fibration  $\omega$ .  $C \wedge \beta$  will serve just fine as a good cylinder object. Hence, we can assume  $C \wedge B' = C \wedge \beta$ .

Now, take a lift

$$\begin{array}{ccccc}
(C \wedge I) \sqcup (C \wedge I) & \xrightarrow{p \sqcup p} & C \sqcup C & \longrightarrow & (C \wedge I) \sqcup_{C \sqcup C} (C \wedge I) & \xrightarrow{\gamma} & C \wedge \beta \\
v \downarrow & & & & \kappa \dashrightarrow & & \downarrow \omega \\
C \wedge I \wedge I & \xrightarrow{k} & & & & & C \wedge B
\end{array}$$

The map  $\kappa$ , together with the natural map  $C \sqcup C \rightarrow C \wedge \beta$  given by  $\gamma$ , induces a map by universality of the pushout  $e : C \wedge B \rightarrow C \wedge \beta$ . Since  $k$  and  $\omega$  are weak equivalences,  $\kappa$  is a weak equivalence. Thus, since  $\kappa$  and  $k$  are weak equivalences,  $e$  must be a weak equivalence. Note that  $e$  commutes with the maps  $C \sqcup C \rightarrow C \wedge \beta$  and  $s : C \sqcup C \rightarrow C \wedge B$ , since  $e$  is induced by the pushout's universal property. Precomposing with  $e$  thus gives the left homotopy  $C \wedge B \rightarrow D$  in  $(C \sqcup C \downarrow \mathcal{M})$  desired, which when precomposed with  $k$  gives  $\Gamma$ .  $\square$

In particular, we find the following:

**Corollary 4.5.1.** *Consider the lifting problem in Corollary 4.2.1, along with the data given in Lemma 4.4. Any two left homotopies  $H, K : C \wedge I \rightarrow B$  between  $\delta$  and  $\epsilon$  induced as in the corollary's methods will be related by a left homotopy*

$$\Gamma : C \wedge I \wedge I \rightarrow B$$

such that  $\Gamma \circ v = (\delta \circ p) + (\epsilon \circ p)$ .

*Proof.*  $H$  and  $K$  are solutions of the same lifting problem. Thus, by Proposition 4.10, there is a left homotopy between them in  $(C \sqcup C \downarrow \mathcal{M})$ , implying the result by Lemma 4.5.  $\square$

Note then that  $\Gamma$ , in our use case, is exactly the definition of a globular 2-homotopy. Thus, we have the following result:

**Theorem 4.6.** *Consider a lifting problem*

$$\begin{array}{ccc} & & B \\ & & \downarrow p \\ C & \xrightarrow{g} & D \end{array}$$

in  $\mathbf{CSSP}$ , where  $C$  is cofibrant and  $p$  is a trivial fibration. Any two solutions will be related by a left homotopy in  $(\mathbf{CSSP} \downarrow D)$  using the cylinder object  $C \times N(I[1])$ , while any two such left homotopies will be related by a globular 2-homotopy.

**Corollary 4.6.1.** *Let  $f, g : C \rightarrow B$  be two solutions to the lifting problem above. Let  $H, K : C \times N(I[1]) \rightarrow B$  be two left homotopies over  $D$  between them. Then the enrichment of  $h_1^c$  in Kan complexes sends  $H$  and  $K$  to the same natural isomorphism  $h_1^c(C) \times I[1] \rightarrow h_1^c(D)$ .*

As an aside, a consequence of our approach to left homotopies that now deserves mentioning is that  $h_1^c$  will send a section of a trivial fibration to its inverse:

**Proposition 4.12.** *Consider a lifting problem of complete Segal spaces*

$$\begin{array}{ccc} & & A \\ & \nearrow g & \downarrow f \\ B & \xrightarrow{id} & B \end{array}$$

with solution  $g$  and a trivial fibration  $f$ . Then  $h_1^c(g)$  is a categorical inverse to  $h_1^c(f)$ .

*Proof.* We have that  $h_1^c(f) \circ h_1^c(g) = 1_{h_1^c(B)}$ . Moreover, by the 2-out-of-3 property of weak equivalences, we have that  $g$  is a Dwyer-Kan equivalence and thus  $h_1^c(g)$  is an equivalence.

The proof will be complete if we produce a left homotopy  $H : A \times N([I]) \rightarrow A$  from  $g \circ f$  to  $1_A$ , as  $h_1^c$  will send this to the necessary natural isomorphism. Such a left homotopy can be produced by noting that  $g \circ f$  and  $1_A$  are both solutions to the lifting problem

$$\begin{array}{ccc} & & A \\ & \nearrow^{g \circ f} & \downarrow f \\ A & \xrightarrow{f} & B \end{array}$$

completing the proof.  $\square$

Our results also make it possible to prove that composition of left homotopies and natural isomorphisms coincide:

**Proposition 4.13.** *In the situation of Definition 4.5, the vertical composite of natural isomorphisms  $h_1^c(K)h_1^c(H)$  is equal to  $h_1^c(KH)$  for any composite  $KH$ .*

*Proof.* Let  $Q$  be as in Proposition 4.5. The composite of  $h_1^c(K)$  and  $h_1^c(H)$  amounts to considering the induced functor

$$F : h_1^c(C) \times I[2] \rightarrow h_1^c(B)$$

which restricts to  $h_1^c(H)$  and  $h_1^c(K)$ . There is precisely one such functor. However,  $h_1^c(Q)$  also shares this property, so  $h_1^c(Q) = F$ . Hence, restriction yields  $h_1^c(KH)$  for any  $KH$  as needed.  $\square$

## 5 Constructing Homotopy Bicategories

Let  $X \in \mathbf{CSSP}_2$  be a complete 2-fold Segal space. We wish to define an unbiased bicategory  $h_2(X)$ . As we will soon find, it will not be reasonable to expect such a bicategory to exist without additional data. This will take the form of a choice of horizontal compositions.

We set  $\mathbf{ob}(h_2(X)) := (X_{0,0})_0$ , for consistency with  $h_1$ . We also define

$$\mathbf{Hom}_{h_2(X)}(x, y) := h_1^c(X(x, y)).$$

We then take a choice of horizontal compositions  $(\mu_n)_{n \geq 0}$  for  $X$  as in Definition 2.42. Composition functors, for  $n \geq 0$  and  $x_0, \dots, x_n \in \mathbf{ob}(h_2(X))$ , will then be the maps

$$\bullet^{x_0, \dots, x_n} : \prod_{i=1}^n \mathbf{Hom}_{h_2(X)}(x_{i-1}, x_i) \cong h_1^c\left(\prod_{i=1}^n X(x_{i-1}, x_i)\right) \xrightarrow{h_1^c(\circ^{x_0, \dots, x_n})} \mathbf{Hom}_{h_2(X)}(x_0, x_n).$$

In general, for a simplicial composition diagram  $K$  of arity  $n$ , write

$$\bullet_K^{x_0, \dots, x_n} : \prod_{i=1}^n \mathbf{Hom}_{h_2(X)}(x_{i-1}, x_i) \cong h_1^c\left(\prod_{i=1}^n X(x_{i-1}, x_i)\right) \xrightarrow{h_1^c(\circ_K^{x_0, \dots, x_n})} \mathbf{Hom}_{h_2(X)}(x_0, x_n).$$

Note the domain is  $*$   $\cong \{x_0\}$  if  $n = 0$ , as with  $\circ^{x_0}$ .

## 5.1 Coherence Isomorphisms

We now must provide the associators and unitors. We will in fact provide much more general machinery here, which will be needed to work with functors as well. For associators, let  $n \in \mathbb{Z}_{>0}$  and  $k_1, \dots, k_n \in \mathbb{Z}_{\geq 0}$ , with  $r = \sum_{i=1}^n k_i$ . Consider any  $n$ -tuple of tuples of elements of  $(X_{0,0})_0$

$$Y := ((x_0^1, x_1^1, \dots, x_{k_1}^1), \dots, (x_0^n, \dots, x_{k_n}^n))$$

such that  $x_{k_i}^i = x_0^{i+1}$  for every  $i < n$ . Let  $(x_0, \dots, x_r)$  be the flattened tuple after removing each  $x_{k_i}^i$  for  $i < n$ .

**Proposition 5.1.** *Let  $Y$  be as above. Let  $Sp(r) \xrightarrow{\iota} K \xleftarrow{\tau} \Delta[1]$  be a simplicial composition diagram where  $K = (K_1 \sqcup_{\Delta[0]} \dots \sqcup_{\Delta[0]} K_n) \sqcup_{Sp(n)} K_0$ , with each diagram  $K_i$  of arity  $k_i$  where  $k_0 = n$ . Then*

$$\bullet_{K_0}^{x_0, \dots, x_r} = \bullet_{K_0}^{x_0^1, \dots, x_0^n, x_{k_1}^n} \circ (\bullet_{K_1}^{x_0^1, \dots, x_{k_1}^1} \times \dots \times \bullet_{K_n}^{x_0^n, \dots, x_{k_n}^n}).$$

*Proof.* We have by Theorem 2.5 that the diagram

$$\begin{array}{ccc} \prod_{i=1}^r \mathbf{Hom}_{h_2(X)}(x_{i-1}, x_i) & \xrightarrow{\cong} & h_1^c(\prod_{i=1}^r X(x_{i-1}, x_i)) \\ \cong \downarrow & \swarrow \cong & \downarrow h_1^c(\circ_K^{x_0, \dots, x_r}) \\ \prod_{i=1}^n h_1^c(\prod_{j=1}^{k_i} X(x_{j-1}^i, x_j^i)) & & \\ \prod_{i=1}^n h_1^c(\circ_{K_i}^{x_0^i, \dots, x_{k_i}^i}) \downarrow & h_1^c(\circ_{K_1}^{x_0^1, \dots, x_{k_1}^1} \times \dots \times \circ_{K_n}^{x_0^n, \dots, x_{k_n}^n}) & \\ \prod_{i=1}^n \mathbf{Hom}_{h_2(X)}(x_0^i, x_{k_i}^i) & & \\ \cong \downarrow & \swarrow & \downarrow \\ h_1^c(\prod_{i=1}^n X(x_0^i, x_{k_i}^i)) & \xrightarrow{h_1^c(\circ_{K_0}^{x_0^1, \dots, x_0^n, x_{k_1}^n})} & \mathbf{Hom}_{h_2(X)}(x_0, x_r) \end{array}$$

commutes. The result follows.  $\square$

**Definition 5.1.** *Let  $X$  be a complete 2-fold Segal space with a choice of horizontal compositions  $(\mu_n)_{n \geq 0}$ . Let  $x, y \in (X_{0,0})_0$  and set  $n \geq 0$ . Define the functor*

$$\omega_{(X, (\mu_n)_{n \geq 0})}^{x, y} : \mathbf{SCD}_n \rightarrow \mathbf{Fun}(h_1^c(X_{Sp(n)}^{x, y}), h_1^c(X(x, y)))$$

to send  $K \mapsto h_1^c((\tau_K^*)^{x, y} \circ \mu_K^{x, y})$  and  $f : K_1 \rightarrow K_2$  to be  $h_1^c((\tau_{K_1}^*)^{x, y} \circ \beta_f^{x, y})$  for an induced left homotopy  $\beta_f$  from  $\mu_{K_1}$  to  $\mu'_{K_2}$ .

**Proposition 5.2.**  $\omega_{(X, (\mu_n)_{n \geq 0})}^{x, y}$  is a functor.

*Proof.* Suppose  $K_1 \xrightarrow{f} K_2 \xrightarrow{g} K_3$  is a diagram in  $\mathbf{SCD}_n$  for simplicial composition diagrams  $\Delta[n] \xrightarrow{\iota_i} K_i \xleftarrow{\tau_i} \Delta[1]$ . We wish to show that

$$\omega_{(X, (\mu_n)_{n \geq 0})}^{x, y}(g) \omega_{(X, (\mu_n)_{n \geq 0})}^{x, y}(f) = \omega_{(X, (\mu_n)_{n \geq 0})}^{x, y}(g \circ f).$$

These three natural isomorphisms can be rewritten as

$$\begin{aligned}
\omega_{(X,(\mu_n)_{n \geq 0})}^{x,y}(f) &= (\tau_1^*)^{x,y} \circ \beta_f^{x,y} \\
\omega_{(X,(\mu_n)_{n \geq 0})}^{x,y}(g) &= (\tau_1^*)^{x,y} \circ (f^* \circ \beta_g)^{x,y} \\
\omega_{(X,(\mu_n)_{n \geq 0})}^{x,y}(g \circ f) &= (\tau_1^*)^{x,y} \circ \beta_{g \circ f}^{x,y}
\end{aligned}$$

for left homotopies  $\beta_f^{x,y}$ ,  $\beta_g^{x,y}$  and  $\beta_{g \circ f}^{x,y}$  induced between  $\mu_{K_1}^{x,y}$  and  $(f^* \circ \mu_{K_2})^{x,y}$ ,  $(f^* \circ \mu_{K_2})^{x,y}$  and  $(f^* \circ g^* \circ \mu_{K_3})^{x,y}$  and  $\mu_{K_1}^{x,y}$  and  $(f^* \circ g^* \circ \mu_{K_3})^{x,y}$  respectively, induced by the evident lifting problems.

We may therefore take a composite  $C = ((f^* \circ \beta_g)^{x,y})(\beta_f^{x,y})$  as in Definition 4.5, which is a left homotopy from  $\mu_{K_1}^{x,y}$  to  $(f^* \circ g^* \circ \mu_{K_3})^{x,y}$ . By Proposition 4.13, we have that  $h_1^c(C)$  is the composite of  $h_1^c((f^* \circ \beta_g)^{x,y})$  and  $h_1^c(\beta_f^{x,y})$ .

Note however that  $C$  and  $\beta_{g \circ f}^{x,y}$  are both left homotopies induced between the same solutions of a lifting problem. Hence, by Corollary 4.6.1, we have that  $h_1^c(\beta_{g \circ f}^{x,y}) = h_1^c(C)$  is the composite of the natural isomorphisms  $h_1^c((f^* \circ \beta_g)^{x,y})$  and  $h_1^c(\beta_f^{x,y})$ , as needed.

Now, suppose  $K$  is a simplicial composition diagram. We wish to show that

$$\omega_{(X,(\mu_n)_{n \geq 0})}^{x,y}(1_K) = 1_{h_1^c((\tau_K^*)^{x,y} \circ \mu_K^{x,y})}.$$

The left homotopy  $\beta_{1_K}$  can be set to be a constant left homotopy, as its source and target are equal. Since any choice of  $\beta_{1_K}$  will yield the same natural isomorphism, the result is evident.  $\square$

**Definition 5.2.** Let  $X$  be a complete 2-fold Segal space with a choice of horizontal compositions  $(\mu_n)_{n \geq 0}$ . Let  $x_0, \dots, x_n \in (X_{0,0})_0$ . Then define the map

$$\phi_{(X,(\mu_n)_{n \geq 0})}^{x_0, \dots, x_n} : \mathbf{Fun}\left(h_1^c(X_{Sp(n)}^{x_0, x_n}), h_1^c(X(x_0, x_n))\right) \rightarrow \mathbf{Fun}\left(\prod_{i=1}^n h_1^c(X(x_{i-1}, x_i)), h_1^c(X(x_0, x_n))\right)$$

to be the functor given by precomposition with the inclusion

$$F_{x_0, \dots, x_n} : \prod_{i=1}^n h_1^c(X(x_{i-1}, x_i)) \cong h_1^c\left(\prod_{i=1}^n X(x_{i-1}, x_i)\right) \hookrightarrow h_1^c(X_{Sp(n)}^{x_0, x_n}).$$

$$\text{Write } \Phi_{(X,(\mu_n)_{n \geq 0})}^{x_0, \dots, x_n} := \phi_{(X,(\mu_n)_{n \geq 0})}^{x_0, \dots, x_n} \circ \omega_{(X,(\mu_n)_{n \geq 0})}^{x_0, x_n}.$$

For our given  $Y$ , let  $K = (\Delta[k_1] \sqcup_{\Delta[0]} \dots \sqcup_{\Delta[0]} \Delta[k_n]) \sqcup_{Sp(n)} \Delta[n]$  and take the map  $f_{k_1, \dots, k_n} : K \rightarrow \Delta[r]$  in  $\mathbf{SCD}_r$  to be the map restricting to the morphisms

$$\begin{aligned}
\left\langle \sum_{j=1}^{i-1} k_j, \dots, \sum_{j=1}^{i-1} k_j + k_i \right\rangle : \Delta[k_i] &\rightarrow \Delta[r] \\
\left\langle 0, k_1, k_1 + k_2, \dots, \sum_{j=1}^n k_j \right\rangle : \Delta[n] &\rightarrow \Delta[r].
\end{aligned}$$

Our associators for  $h_2$  will be the natural isomorphisms

$$\Phi_{(X,(\mu_n)_{n \geq 0})}^{x_0, \dots, x_r}(f_{k_1, \dots, k_n}) : \bullet_{x_0^1, \dots, x_0^n, x_n^n} \circ (\bullet_{x_0^1, \dots, x_{k_1}^1} \times \dots \times \bullet_{x_0^n, \dots, x_{k_n}^n}) \Rightarrow \bullet_{x_0^1, \dots, x_r}.$$

Unitors for us will be trivial; we choose to set them to be identities. We are thus now ready to declare the definition of our unbiased bicategory in full:

**Definition 5.3.** Let  $X$  be a complete 2-fold Segal space. Let  $(\mu_n)_{n \geq 0}$  be a choice of horizontal compositions.

Then the unbiased homotopy bicategory  $h_2(X, (\mu_n)_{n \geq 0})$  of  $X$ , written as  $h_2(X)$  if the  $\mu_n$  are known, is the unbiased bicategory defined such that:

1.  $\mathbf{ob}(h_2(X)) := (X_{0,0})_0$ ;
2.  $\mathbf{Hom}_{h_2(X)}(x, y) := h_1^c(X(x, y))$  for all  $x, y \in (X_{0,0})_0$ ;
3. For each  $n > 0$  and  $x_0, \dots, x_n \in (X_{0,0})_0$ , composition is given by the functor  $\bullet^{x_0, \dots, x_n}$ ;
4. For each  $x \in (X_{0,0})_0$ , identities are given by the functor  $\bullet^x$ ;
5. Let  $n > 0$  and  $k_1, \dots, k_n \in \mathbb{Z}_{\geq 0}$  with  $r = \sum_i k_i$ . Consider any  $n$ -tuple of tuples of elements of  $(X_{0,0})_0$

$$Y := ((x_0^1, x_1^1, \dots, x_{k_1}^1), \dots, (x_0^n, \dots, x_{k_n}^n))$$

such that  $x_{k_i}^i = x_0^{i+1}$  for every  $i$ . Let  $(x_0, \dots, x_r)$  be the flattened tuple after removing each  $x_{k_i}^i$  for  $i < n$ . Then the natural isomorphism  $\gamma_Y$  is defined to be

$$\Phi_{(X, (\mu_n)_{n \geq 0})}^{x_0, \dots, x_r}(f_{k_1, \dots, k_n});$$

6. For each  $x, y \in (X_{0,0})_0$ , the natural isomorphism

$$\iota_{x,y} : \mathbf{1}_{\mathbf{Hom}_{h_2(X)}(x,x)} \Rightarrow \mathbf{1}_{\mathbf{Hom}_{h_2(X)}(x,x)} = \bullet^{x,x}$$

is an identity.

## 5.2 Coherence Conditions

Our attention turns at this point to proving that we have indeed specified a valid unbiased bicategory in Definition 5.3. A technical lemma is first required:

**Lemma 5.1.** Consider a diagram in a model category  $\mathcal{M}$

$$\begin{array}{ccccccc}
 & & & & E & \xrightarrow{s} & F & \xrightarrow{w} & Z \\
 & & & & q \downarrow & \nearrow \epsilon & \downarrow r & & \\
 A & \xrightarrow{h} & B & \xrightarrow{u} & G & \xrightarrow{t} & H & & \\
 f \downarrow & \nearrow \delta & \downarrow g & & & & & & \\
 C & \xrightarrow{k} & D & & & & & & 
 \end{array}$$

where the two squares define lifting problems with solutions  $\delta$  and  $\epsilon$ , such that  $g$  and  $r$  are trivial fibrations. Then the square in the diagram

$$\begin{array}{ccccc}
 A & \xrightarrow{h \times_t u h \epsilon u h} & B \times_H F & \twoheadrightarrow & F & \xrightarrow{w} & Z \\
 f \downarrow & \nearrow \delta \times_u \delta \epsilon u \delta & \downarrow & & \downarrow & & \\
 C & \xrightarrow{k} & D & & B & \xrightarrow{g} & 
 \end{array}$$

is also a lifting problem in  $\mathcal{M}$  with the given solution. Moreover, the map  $C \rightarrow B \times_H F \rightarrow F \rightarrow Z$  in this diagram is equal to  $weu\delta$ .

*Proof.* It is clear that the rightmost map will be a trivial fibration if  $g$  and  $r$  are as such. Thus, the diagram is a valid lifting problem. The last equality is easily checked by the definition of the chosen lift.  $\square$

The following lemma allows us to compose left homotopies horizontally in an ‘operadic’ manner, which will be necessary for our coherence condition:

**Lemma 5.2.** *Let  $X$  be a complete Segal space and set  $n \geq 1$  and  $k_1, \dots, k_n \geq 0$ . Take complete Segal spaces  $A_i^j$  for  $1 \leq i \leq n$  and  $1 \leq j \leq k_i$ , complete Segal spaces  $B_i$  and  $B'_i$  for  $1 \leq i \leq n$  and some complete Segal space  $C$ , all fibered over  $X^2$ . Take between these a set of maps in CSSP for  $1 \leq i \leq n$  of the form*

$$\begin{aligned} f_i, g_i &: \prod_X^{1 \leq j \leq k_i} A_i^j \rightarrow B_i \\ t_i &: B_i \rightarrow B'_i \\ \phi, \psi &: \prod_X^{1 \leq i \leq n} B'_i \rightarrow C. \end{aligned}$$

Consider a finite collection of left homotopies in CSSP emerging from lifting problems in commutative diagrams fibered over  $X^2$

$$\begin{array}{ccc} \prod_X^{1 \leq j \leq k_i} A_i^j \sqcup \prod_X^{1 \leq j \leq k_i} A_i^j & \xrightarrow{f_i \sqcup g_i} & B_i \xrightarrow{t_i} B'_i \\ \downarrow & \dashrightarrow^{H_i} & \downarrow \\ \prod_X^{1 \leq j \leq k_i} A_i^j \times N(I[1]) & \longrightarrow & \prod_X^{1 \leq j \leq k_i} A_i^j \end{array}$$

for  $1 \leq i \leq n$  where the right vertical map is a trivial fibration and

$$\begin{array}{ccc} \prod_X^{1 \leq i \leq n} B'_i \sqcup \prod_X^{1 \leq i \leq n} B'_i & \xrightarrow{\psi \sqcup \phi} & C \\ \downarrow & \dashrightarrow^K & \downarrow \\ \prod_X^{1 \leq i \leq n} B'_i \times N(I[1]) & \longrightarrow & \prod_X^{1 \leq i \leq n} B'_i \end{array}$$

where again the right vertical map is a trivial fibration. For all  $k \geq 1$ , set  $D_k : \prod_X^{i,j} A_i^j \times N(I[1]) \rightarrow \prod_X^{i,j} A_i^j \times N(I[1])^k$  to be the diagonal map.

Then the left homotopy

$$Q := K \circ \left( \left( \prod_X^{1 \leq i \leq n} t_i \circ H_i \right) \times 1_{N(I[1])} \right) \circ D_2$$

is equal to  $p \circ F$ , where  $p$  and  $F$  are fibered naturally over  $X^2$  and fit into a lifting problem

in *CSSP*

$$\begin{array}{ccc}
\prod_X^{i,j} A_i^j \sqcup \prod_X^{i,j} A_i^j & \xrightarrow{\iota} & \prod_X^i B_i \times \prod_X^{i, B'_i} C \xrightarrow{p} C \\
\downarrow & \nearrow F & \downarrow \\
\prod_X^{i,j} A_i^j \times N(I[1]) & \longrightarrow & \prod_X^{i,j} A_i^j
\end{array}$$

such that  $p$  is the pullback projection and  $\iota$  is induced by the maps

$$\prod_X^{i,j} A_i^j \sqcup \prod_X^{i,j} A_i^j \xrightarrow{\prod_X^i f_i \sqcup \prod_X^i g_i} \prod_X^i B_i$$

and

$$\prod_X^{i,j} A_i^j \sqcup \prod_X^{i,j} A_i^j \xrightarrow{\prod_X^i f_i + \prod_X^i g_i} \prod_X^i B_i \sqcup \prod_X^i B_i \xrightarrow{\prod_X^i t_i + \prod_X^i t_i} \prod_X^i B'_i \sqcup \prod_X^i B'_i \xrightarrow{\psi \sqcup \phi} C.$$

*Proof.* Define

$$q : \prod_X^{i,j} A_i^j \sqcup \prod_X^{i,j} A_i^j \hookrightarrow \prod_X^{1 \leq i \leq n} \left( \prod_X^j A_i^j \sqcup \prod_X^j A_i^j \right) \cong \bigsqcup_{k=1}^{2^n} \prod_X^{i,j} A_i^j$$

to be the pullback map induced by the  $n$  maps  $\rho_i \sqcup \rho_i$  for  $1 \leq i \leq n$ , where  $\rho_i : \prod_X^{i,j} A_i^j \rightarrow \prod_X^j A_i^j$  is the projection to the  $i^{\text{th}}$  coordinate. This is implicitly an inclusion.

We have that  $(\prod_X^i H_i) \circ D_n$  is a solution of the lifting problem defined by the outermost square of the diagram

$$\begin{array}{ccc}
\prod_X^{i,j} A_i^j \sqcup \prod_X^{i,j} A_i^j & \xrightarrow{\prod_X^{1 \leq i \leq n} f_i \sqcup \prod_X^{1 \leq i \leq n} g_i} & \prod_X^{1 \leq i \leq n} B_i \\
\downarrow q & \searrow & \downarrow \\
\prod_X^{1 \leq i \leq n} (\prod_X^j A_i^j \sqcup \prod_X^j A_i^j) & \xrightarrow{\prod_X^{1 \leq i \leq n} (f_i \sqcup g_i)} & \prod_X^{1 \leq i \leq n} B_i \\
\downarrow & \nearrow \text{dashed } \prod_X^{1 \leq i \leq n} H_i & \downarrow \\
\prod_X^{1 \leq i \leq n} (\prod_X^j A_i^j \times N(I[1])) & \longrightarrow & \prod_X^{i,j} A_i^j \\
\downarrow & \nearrow & \\
\prod_X^{i,j} A_i^j \times N(I[1]) & & 
\end{array}$$

which demonstrates  $(\prod_X^i H_i) \circ D_n$  is an induced left homotopy from  $\prod_X^i f_i$  to  $\prod_X^i g_i$ .



We now have a commutative diagram of the form

$$\begin{array}{ccc}
& \prod_X^{1 \leq i \leq n} B'_i \sqcup \prod_X^{1 \leq i \leq n} B'_i & \xrightarrow{\psi \sqcup \phi} & C \\
& \downarrow & \nearrow K & \downarrow \\
& \prod_X^{1 \leq i \leq n} B'_i \times N(I[1]) & \longrightarrow & \prod_X^{1 \leq i \leq n} B'_i \\
& \uparrow \prod_X^i t_i \times 1_{N(I[1])} & & \\
(\prod_X^{i,j} A_i^j \sqcup \prod_X^{i,j} A_i^j) \times N(I[1]) & \xrightarrow{(\prod_X^i f_i \sqcup \prod_X^i g_i) \times 1_{N(I[1])}} & \prod_X^{1 \leq i \leq n} B_i \times N(I[1]) & \\
\downarrow & \dashrightarrow (\prod_X^i H_i) \times 1_{N(I[1])} & \downarrow & \\
\prod_X^{i,j} A_i^j \times N(I[1])^{n+1} & \longrightarrow & \prod_X^{i,j} A_i^j \times N(I[1]) & \\
\downarrow & \nearrow & & \\
\prod_X^{i,j} A_i^j \times N(I[1])^2 & & & 
\end{array}$$

which, by Lemma 5.1, produces a single lifting problem with solution  $F'$  within the diagram

$$\begin{array}{ccc}
\prod_X^{i,j} A_i^j \sqcup \prod_X^{i,j} A_i^j & \xrightarrow{\iota'} & C \\
\downarrow (id, \langle 0 \rangle) + (id, \langle 1 \rangle) & & \downarrow p \\
(\prod_X^{i,j} A_i^j \sqcup \prod_X^{i,j} A_i^j) \times N(I[1]) & \longrightarrow & \left( \prod_X^i B_i \times N(I[1]) \right) \times_{\prod_X B'_i} C \xrightarrow{R} \prod_X^i B_i \times_{\prod_X B'_i} C \\
\downarrow & \dashrightarrow F' & \downarrow \\
\prod_X^{i,j} A_i^j \times N(I[1])^2 & \longrightarrow & \prod_X^{i,j} A_i^j \times N(I[1]) \\
\downarrow & & \downarrow \\
\prod_X^{i,j} A_i^j \times N(I[1]) & \longrightarrow & \prod_X^{i,j} A_i^j
\end{array}$$

where the maps  $\langle 0 \rangle, \langle 1 \rangle : * \rightarrow N(I[1])$  identify the two objects of  $I[1]$  and  $R$  is induced by the projection forgetting  $N(I[1])$ . The outermost square of this diagram is another lifting problem, solved by  $F := R \circ F' \circ D_2$ . It is clear then that  $p \circ F = Q$  by design. Moreover, since the diagram

$$\begin{array}{ccc}
\prod_X^{i,j} A_i^j \sqcup \prod_X^{i,j} A_i^j & \xrightarrow{\prod_X^i f_i + \prod_X^i g_i} & \prod_X^i B_i \sqcup \prod_X^i B_i \\
\downarrow (id, \langle 0 \rangle) + (id, \langle 1 \rangle) & & \downarrow \prod_X^i t_i + \prod_X^i t_i \\
\left( \prod_X^{i,j} A_i^j \sqcup \prod_X^{i,j} A_i^j \right) \times N(I[1]) & & \prod_X^i B'_i \sqcup \prod_X^i B'_i \\
\downarrow (\prod_X^i f_i \sqcup \prod_X^i g_i) \times 1_{N(I[1])} & & \downarrow \psi \sqcup \phi \\
\left( \prod_X^i B_i \right) \times N(I[1]) & & \\
\downarrow (\prod_X^i t_i) \times 1_{N(I[1])} & & \\
\left( \prod_X^i B'_i \right) \times N(I[1]) & \xrightarrow{K} & C
\end{array}$$

commutes, setting  $\iota := R \circ \iota'$  completes the proof.  $\square$

A crucial and nontrivial lemma is now needed, which carries a remarkably operadic flavor to it. We will not explore any subtle appearances of operads in this paper; such concerns will be left to future work. One could argue this lemma is the reason we have coherence:

**Lemma 5.3.** *Let  $X$  and the list  $Y$  be as above, with some choice of horizontal compositions  $(\mu_n)_{n \geq 0}$ . Consider the functor  $(-) \cdot (-)$  of the form*

$$\begin{aligned} \mathbf{Fun}\left(\prod_{i=1}^n h_1^c(X(x_0^i, x_{k_i}^i)), h_1^c(X(x_0, x_r))\right) \times \mathbf{Fun}\left(\prod_{i=1}^r h_1^c(X(x_{i-1}, x_i)), \prod_{i=1}^n h_1^c(X(x_0^i, x_{k_i}^i))\right) \\ \downarrow (-) \cdot (-) \\ \mathbf{Fun}\left(\prod_{i=1}^r h_1^c(X(x_{i-1}, x_i)), h_1^c(X(x_0, x_r))\right) \end{aligned}$$

defined by horizontal composition. Then

$$\Phi_{(X, (\mu_n)_{n \geq 0})}^{x_0^1, \dots, x_0^n, x_{k_n}^n} \cdot \left( \prod_{i=1}^n \Phi_{(X, (\mu_n)_{n \geq 0})}^{x_0^i, \dots, x_{k_i}^i} \right) = \Phi_{(X, (\mu_n)_{n \geq 0})}^{x_0, \dots, x_r} \circ \mathcal{G}_{k_1, \dots, k_n}^n.$$

*Proof.* That this equivalence holds on objects is by construction. For morphisms, consider for  $0 \leq i \leq n$  a collection of maps  $f_i : K_i \rightarrow K'_i$  in  $\mathbf{SCD}_{k_i}$ , where  $\Delta[k_i] \xrightarrow{\iota_i} K_i \xleftarrow{\tau_i} \Delta[1]$  and  $\Delta[k_i] \xrightarrow{\iota'_i} K'_i \xleftarrow{\tau'_i} \Delta[1]$  are simplicial composition diagrams and  $k_0 = n$ . Then we have an induced map  $f : K \rightarrow K'$ , where

$$\begin{aligned} K &:= (K_1 \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} K_n) \sqcup_{Sp(n)} K_0 \\ K' &:= (K'_1 \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} K'_n) \sqcup_{Sp(n)} K'_0 \end{aligned}$$

with maps  $\iota, \tau, \iota'$  and  $\tau'$ . We wish to show that

$$\Phi_{(X, (\mu_n)_{n \geq 0})}^{x_0^1, \dots, x_0^n, x_{k_n}^n}(f_0) \circ \prod_{i=1}^n \Phi_{(X, (\mu_n)_{n \geq 0})}^{x_0^i, \dots, x_{k_i}^i}(f_i) = \Phi_{(X, (\mu_n)_{n \geq 0})}^{x_0, \dots, x_r}(f).$$

The left hand side of this equality can be expanded out as

$$h_1^c((\tau_0^*)^{x_0^1, x_{k_n}^n}) \circ h_1^c(\beta_{f_0}^{x_0^1, x_{k_n}^n}) \circ F_{x_0^1, \dots, x_0^n, x_{k_n}^n} \circ \left( \prod_{i=1}^n h_1^c((\tau_i^*)^{x_0^i, x_{k_i}^i}) \circ h_1^c(\beta_{f_i}^{x_0^i, x_{k_i}^i}) \circ F_{x_0^i, \dots, x_{k_i}^i} \right)$$

where each  $\beta_{f_i}$  is a left homotopy induced between  $\mu_{K_i}$  and  $\mu'_{K'_i}$ . The right hand side can then be expanded out as

$$h_1^c((\tau^*)^{x_0, x_r}) \circ h_1^c(\beta_f^{x_0, x_r}) \circ F_{x_0, \dots, x_r}.$$

If we can show that the left hand side is also of the form  $h_1^c((\tau^*)^{x_0^1, x_{k_n}^n}) \circ h_1^c(H^{x_0, x_r}) \circ F_{x_0, \dots, x_r}$  for some left homotopy  $H$  induced between  $\mu_K$  and  $\mu'_{K'}$ , the proof will be complete, as these will then have to be identified by Corollary 4.6.1.

We should note that, if we are to treat these natural isomorphisms as functors of the form  $\mathcal{C} \times I[1] \rightarrow \mathcal{D}$ , we will have to rewrite the left-hand side as

$$h_1^c((\tau_0^*)^{x_0^1, x_{k_n}^n} \circ \beta_{f_0}^{x_0^1, x_{k_n}^n}) \circ F'_{x_0^1, \dots, x_0^n, x_{k_n}^n} \circ \left( \left( \prod_{i=1}^n h_1^c((\tau_i^*)^{x_0^i, x_{k_i}^i} \circ \beta_{f_i}^{x_0^i, x_{k_i}^i}) \circ F'_{x_0^i, \dots, x_{k_i}^i} \circ D'_n \right) \times 1_{I[1]} \right) \circ D'_2$$

where we now have the inclusion

$$F'_{a_0, \dots, a_k} : \prod_{i=1}^k h_1^c(X(a_{i-1}, a_i)) \times I[1] \hookrightarrow h_1^c(X_{Sp(k)}^{a_0, a_k} \times N(I[1]))$$

and  $D'_k : \prod_{i=1}^r \mathbf{Hom}_{h_2(X)}(x_{i-1}, x_i) \times I[1] \rightarrow \prod_{i=1}^r \mathbf{Hom}_{h_2(X)}(x_{i-1}, x_i) \times I[1]^k$  the diagonal map for all  $k > 0$ .

Using naturality of the isomorphism  $h_1^c(-) \times h_1^c(-) \cong h_1^c(- \times -)$ , we can convert this into a map  $h_1^c((\tau_0^*)^{x_0^1, x_{k_n}^n} \circ P)$ , where

$$P := \beta_{f_0}^{x_0^1, x_{k_n}^n} \circ \mathcal{F}_{x_0^1, \dots, x_0^n, x_{k_n}^n} \circ \left( \left( \prod_{i=1}^n (\tau_i^*)^{x_0^i, x_{k_i}^i} \circ \beta_{f_i}^{x_0^i, x_{k_i}^i} \circ \mathcal{F}_{x_0^i, \dots, x_{k_i}^i} \circ D_n^{x_0, \dots, x_r} \right) \times 1_{N(I[1])} \right) \circ D_2^{x_0, \dots, x_r}$$

with the inclusions

$$\mathcal{F}_{a_0, \dots, a_k} : \prod_{i=1}^k X(a_{i-1}, a_i) \rightarrow X_{Sp(k)}^{a_0, a_k}$$

and the diagonal maps  $D_k : X_{Sp(r)} \times N(I[1]) \rightarrow X_{Sp(r)} \times N([1])^k$  on  $N(I[1])$ .

Note that the maps  $\mathcal{F}_{a_0, \dots, a_k}$  form part of a more general natural transformation  $(-)^{a_0, \dots, a_k} \Rightarrow (-)^{a_0, a_k}$ . Using this naturality, we can prove that  $P$  is in fact equal to a map  $Q^{x_0, x_r} \circ \mathcal{F}_{x_0, \dots, x_r}$  such that

$$Q := \beta_{f_0} \circ \left( \left( \prod_{\substack{1 \leq i \leq n \\ X_0}} \tau_i^* \circ \beta_{f_i} \right) \circ D_n \right) \times 1_{N(I[1])} \circ D_2.$$

This precisely places us in the situation of Lemma 5.2. Note however that the resulting induced left homotopy is of the form

$$H : X_{Sp(r)} \times N(I[1]) \rightarrow X_K$$

and maps from  $\mu_K$  to  $\mu'_{K'}$ . Moreover, given the inclusion  $b : K_0 \hookrightarrow K$ , it is such that  $b^* \circ H = Q$ . Since  $\tau_0^* \circ b^* = \tau^*$ , we have our left homotopy  $H$  such that our original natural isomorphism is of the form

$$h_1^c(\tau^{x_0, x_r}) \circ h_1^c(H^{x_0, x_r}) \circ F_{x_0, \dots, x_r}.$$

This confirms by Corollary 4.6.1 that the equality holds.  $\square$

**Theorem 5.4.** *Let  $X$  be a complete 2-fold Segal space and  $(\mu_n)_{n \geq 0}$  a choice of horizontal compositions. Then  $h_2(X, (\mu_n)_{n \geq 0})$  is an unbiased bicategory.*

*Proof.* We begin by dealing with associativity conditions. Let  $n, m_1, \dots, m_n \in \mathbb{Z}_{>0}$  and  $k_1^1, \dots, k_n^{m_n} \in \mathbb{Z}_{\geq 0}$ . Take a thrice-nested tuple of elements of  $(X_{0,0})_0$  of the form

$$L = (L_p)_{p=1}^n = ((L_{p,q})_{q=1}^{m_p})_{p=1}^n = (((x_{p,q,s})_{s=0}^{k_p^q})_{q=1}^{m_p})_{p=1}^n$$

where  $x_{p,q,k_p^q} = x_{p,q+1,0}$  if  $q < m_p$  and  $x_{p,m_p,k_p^{m_p}} = x_{p+1,1,0}$  if  $p < n$ . Let  $t_p = \sum_{q=1}^{m_p} k_p^q$  and  $r = \sum_{p=1}^n t_p$ . Set  $(x_0, \dots, x_r)$  to be the flattened version of  $L$  after removing the elements  $x_{p,q,k_p^q}$ , not including  $x_{n,m_n,k_n^{m_n}}$ . Let  $(x_0^p, \dots, x_{t_p}^p)$  be the flattened version of  $L_p$  after removing each  $x_{p,q,k_p^q}$  not including  $x_{p,m_p,k_p^{m_p}}$ .

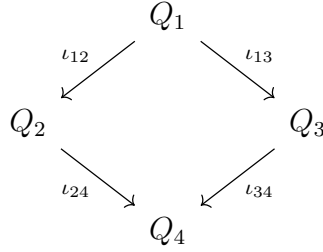
We take an interest in the four composition operations  $\bullet_{Q_i}^{x_0, \dots, x_r}$  for  $i \in \{1, 2, 3, 4\}$ , where the four simplicial composition diagrams  $Q_i$  are as follows:

$$\begin{aligned} Q_1 &:= \left( Q_{11} \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} Q_{1n} \right) \sqcup_{Sp(n)} \Delta[n] \\ Q_2 &:= \left( \Delta[t_1] \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} \Delta[t_n] \right) \sqcup_{Sp(n)} \Delta[n] \\ Q_3 &:= \left( \Delta[k_1^1] \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} \Delta[k_n^{m_n}] \right) \sqcup_{Sp(\sum_{i=1}^n m_i)} \Delta\left[\sum_{i=1}^n m_i\right] \\ Q_4 &:= \Delta[r] \end{aligned}$$

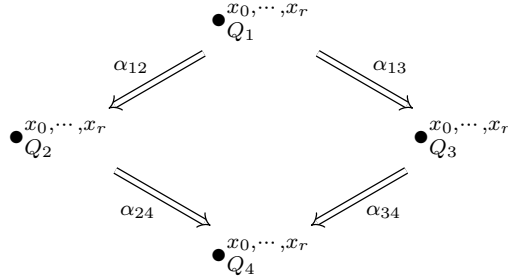
where, for  $1 \leq p \leq n$ ,

$$Q_{1p} := \left( \Delta[k_p^1] \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} \Delta[k_p^{m_p}] \right) \sqcup_{Sp(m_p)} \Delta[m_p].$$

There is a clear diagram in  $\mathbf{SCD}_r$



We wish to show that this induces the diagram of natural isomorphisms



that we seek to prove commutes, where

$$\begin{aligned} \alpha_{12} &:= 1_{\bullet_{x_{1,1,0}, \dots, x_{n,1,0}, x_{n,m_n, k_n^{m_n}}}} \circ \prod_{p=1}^n \Phi_{(X, (\mu_n)_{n \geq 0})}^{x_0^p, \dots, x_{t_p}^p} (f_{k_1^p, \dots, k_{m_p}^p}) \\ \alpha_{13} &:= \Phi_{(X, (\mu_n)_{n \geq 0})}^{x_{1,1,0}, \dots, x_{n,1,0}, x_{n,m_n, k_n^{m_n}}} (f_{m_1, \dots, m_n}) \circ \prod_{p=1}^n 1_{\left( \bullet_{Q_{1p}}^{x_0^p, \dots, x_{t_p}^p} \right)} \\ \alpha_{24} &:= \Phi_{(X, (\mu_n)_{n \geq 0})}^{x_0, \dots, x_r} (f_{t_1, \dots, t_n}) \\ \alpha_{34} &:= \Phi_{(X, (\mu_n)_{n \geq 0})}^{x_0, \dots, x_r} (f_{k_1^1, \dots, k_n^{m_n}}). \end{aligned}$$

Using Lemma 5.3, it is straightforward to identify each  $\alpha_{ij}$  with the corresponding  $\Phi_{(X, (\mu_n)_{n \geq 0})}^{x_0, \dots, x_r} (l_{ij})$ . Thus, by functoriality, the diagram commutes as needed.

The conditions on unitors is then trivial, since all involved natural isomorphisms are identities. Indeed, the relevant associators are  $\Phi_{(X, (\mu_n)_{n \geq 0})}^{x_0, \dots, x_n} (f_{1, \dots, 1})$ , which are constant as  $f_{1, \dots, 1} = 1_{\Delta[n]}$ .  $\square$

### 5.3 Functoriality

In order for  $h_2$  to be a genuine functor, we need to demonstrate how to convert a map  $f : (X, (\mu_n)_{n \geq 0}) \rightarrow (Y, (\nu_n)_{n \geq 0})$  in  $\mathbf{CSSP}_2^{comp}$  into a pseudofunctor

$$h_2(f) : h_2(X, (\mu_n)_{n \geq 0}) \rightarrow (Y, (\nu_n)_{n \geq 0}).$$

Before we can do so, a great deal of the technology we developed for  $h_2$ 's behavior on objects now needs to be extended to handle maps  $X \rightarrow Y$  of complete 2-fold Segal spaces.

We will first need to extend the notion of being object-fibered to a map between complete 2-fold Segal spaces:

**Proposition 5.3.** *Let  $F : X \rightarrow Y$  be a map in  $\mathbf{sSpace}_k$ . Consider two commutative diagrams in  $\mathbf{sSpace}_{k-1}$*

$$\begin{array}{ccc} A & \xrightarrow{F_A} & K \\ \downarrow f & \searrow & \downarrow \\ (X_0)^n & \xrightarrow{(F_0)^n} & (Y_0)^n \\ \uparrow & \nearrow & \uparrow \\ B & \xrightarrow{F_B} & L \end{array} \quad \begin{array}{ccc} A & \xrightarrow{F_A} & K \\ \downarrow & \searrow & \downarrow g \\ (X_0)^n & \xrightarrow{(F_0)^n} & (Y_0)^n \\ \uparrow & \nearrow & \uparrow \\ B & \xrightarrow{F_B} & L \end{array}$$

for some chosen maps  $F_A$  and  $F_B$ , so that  $f$  and  $g$  are object-fibered. Let  $x_0, \dots, x_n \in (X_{0, \dots, 0})_0$ .

Then, defining  $F_B^{x_0, \dots, x_n}$  to be the map

$$\begin{array}{ccc} B^{x_0, \dots, x_n} & \xrightarrow{F_B^{x_0, \dots, x_n}} & L^{F(x_0), \dots, F(x_n)} \\ \downarrow & \nearrow & \\ B^{F(x_0), \dots, F(x_n)} & & \end{array}$$

and similarly for  $F_A^{x_0, \dots, x_n}$ , we have that

$$F_B^{x_0, \dots, x_n} \circ f^{x_0, \dots, x_n} = (F_B \circ f)^{f(x_0), \dots, f(x_n)} \circ \mathcal{I}_{x_0, \dots, x_n}^F,$$

where  $\mathcal{I}_{x_0, \dots, x_n}^{F,A} : A^{x_0, \dots, x_n} \hookrightarrow A^{f(x_0), \dots, f(x_n)}$  is the inclusion. Similarly, we have

$$g^{f(x_0), \dots, f(x_n)} \circ F_A^{x_0, \dots, x_n} = (g \circ F_A)^{f(x_0), \dots, f(x_n)} \circ \mathcal{I}_{x_0, \dots, x_n}^F.$$

*Proof.* Diagram chases show both statements to be true. □

The central idea controlling our composition, coherence isomorphisms and coherence conditions was the collections of functors  $\Phi_{(X, (\mu_n)_{n \geq 0})}^{x_0, \dots, x_n}$  for all  $x_0, \dots, x_n \in (X_{0,0})_0$  and  $n \geq 0$ . The flavor of results we will need to prove for pseudofunctors are similar, so we seek an augmented version of this construct.

In essence, what  $\Phi_{(X, (\mu_n)_{n \geq 0})}^{x_0, \dots, x_n}$  accomplished was to convert diagrams in  $\mathbf{SCD}_n$  into commutative diagrams of coherence isomorphisms. A reasonable extension of this category for the purposes of pseudofunctors is  $\mathbf{SCD}_n \times [1]$ , with the morphisms  $(1_K, 0 < 1)$  representing a homotopy from composition in the codomain to the domain:

**Definition 5.4.** Let  $f : (X, (\mu_n)_{n \geq 0}) \rightarrow (Y, (\nu_n)_{n \geq 0})$  be a map in  $\mathbf{CSSP}_2^{\text{comp}}$ . Let  $x, y \in (X_{0,0})_0$ . Then define the functor

$$\omega_f^{x,y} : \mathbf{SCD}_n \times [1] \rightarrow \mathbf{Fun}\left(h_1^c(X_{Sp(n)}^{x,y}), h_1^c(Y(x, y))\right)$$

to be the functor sending

$$\begin{aligned} (K, 0) &\mapsto h_1^c((\tau_K^*)^{f(x), f(y)} \circ \nu_K^{f(x), f(y)} \circ f_{Sp(n)}^{x,y}) \\ (K, 1) &\mapsto h_1^c((\tau_K^*)^{f(x), f(y)} \circ f_K^{x,y} \circ \mu_K^{x,y}) \end{aligned}$$

and sending a morphism to the map generated by the following cases:

1.  $(g, 1_0) \mapsto \omega_{(Y, (\nu_n)_{n \geq 0})}^{f(x), f(y)}(g) \circ h_1^c(f_{Sp(n)}^{x,y})$ , where  $g : K_1 \rightarrow K_2$ ;
2.  $(g, 1_1) \mapsto h_1^c(f_1^{x,y}) \circ \omega_{(X, (\mu_n)_{n \geq 0})}^{x,y}(g)$ , where  $g$  is as above;
3.  $(1_K, 0 < 1) \mapsto h_1^c((\tau_K^*)^{f(x), f(y)}) \circ h_1^c(\eta_K^{f(x), f(y)}) \circ h_1^c(\mathcal{I}_{x,y}^{f, X_{Sp(n)}})$ , where  $\eta_K^{x,y}$  is an induced left homotopy from  $\nu_K \circ f_{Sp(n)}$  to  $f_K \circ \mu_K$ .

**Proposition 5.4.**  $\omega_f^{x,y}$  is a functor.

*Proof.* Functoriality on the subcategory  $\mathbf{SCD}_n \times \{0, 1\}$  is clear. We need only prove that, for a map  $g : K_1 \rightarrow K_2$ ,

$$\omega_f^{x,y}(g, 1_1) \omega_f^{x,y}(1_{K_1}, 0 < 1) = \omega_f^{x,y}(1_{K_2}, 0 < 1) \omega_f^{x,y}(g, 1_0).$$

Expanding out terms, we have that

$$\begin{aligned} \omega_f^{x,y}(g, 1_0) &= h_1^c\left((\tau_{K_1}^*)^{f(x), f(y)} \circ \alpha_g^{f(x), f(y)} \circ (f_{Sp(n)} \times 1_{N(I[1])})^{f(x), f(y)} \circ \mathcal{I}_{x,y}^{f, X_{Sp(n)} \times N(I[1])}\right) \\ \omega_f^{x,y}(1_{K_1}, 0 < 1) &= h_1^c\left((\tau_{K_1}^*)^{f(x), f(y)} \circ \eta_{K_1}^{f(x), f(y)} \circ \mathcal{I}_{x,y}^{f, X_{Sp(n)} \times N(I[1])}\right) \\ \omega_f^{x,y}(1_{K_2}, 0 < 1) &= h_1^c\left((\tau_{K_2}^*)^{f(x), f(y)} \circ \eta_{K_2}^{f(x), f(y)} \circ \mathcal{I}_{x,y}^{f, X_{Sp(n)} \times N(I[1])}\right) \\ \omega_f^{x,y}(g, 1_1) &= h_1^c\left(f_1^{f(x), f(y)} \circ (\tau_{K_1}^*)^{f(x), f(y)} \circ \beta_g^{f(x), f(y)} \circ \mathcal{I}_{x,y}^{f, X_{Sp(n)} \times N(I[1])}\right) \end{aligned}$$

for induced left homotopies  $\alpha_g$  from  $\nu_{K_1}$  to  $\nu'_{K_2}$  and  $\beta_g$  from  $\mu_{K_1}$  to  $\mu'_{K_2}$ .

We are able to produce the two compositions  $(f_{K_1}^{f(x), f(y)} \circ \beta_g^{f(x), f(y)}) (\eta_{K_1}^{f(x), f(y)})$  and  $((g^*)^{f(x), f(y)} \circ \eta_{K_2}^{f(x), f(y)}) (\alpha_g^{f(x), f(y)} \circ (f_{Sp(n)}^{f(x), f(y)} \times 1_{N(I[1])}))$ , which yield the same natural isomorphisms as before. As these compositions are left homotopies induced between the same solutions to the same lifting problem, the equality holds.  $\square$

**Definition 5.5.** Let  $f : (X, (\mu_n)_{n \geq 0}) \rightarrow (Y, (\nu_n)_{n \geq 0})$  be a map in  $\mathbf{CSSP}_2^{\text{comp}}$ . Let  $x_0, \dots, x_n \in (X_{0,0})_0$ . Then define the map

$$\phi_f^{x_0, \dots, x_n} : \mathbf{Fun}\left(h_1^c(X_{Sp(n)}^{x_0, x_n}), h_1^c(Y(x_0, x_n))\right) \rightarrow \mathbf{Fun}\left(\prod_{i=1}^n h_1^c(X(x_{i-1}, x_i)), h_1^c(Y(x_0, x_n))\right)$$

to be precomposition with the inclusion

$$F_{x_0, \dots, x_n} : \prod_{i=1}^n h_1^c(X(x_{i-1}, x_i)) \cong h_1^c\left(\prod_{i=1}^n X(x_{i-1}, x_i)\right) \hookrightarrow h_1^c(X_{Sp(n)}^{x_0, x_n}).$$

Write  $\Phi_f^{x_0, \dots, x_n} := \phi_f^{x_0, \dots, x_n} \circ \omega_f^{x_0, x_n}$ .

Henceforth, for a map  $f : (X, (\mu_n)_{n \geq 0}) \rightarrow (Y, (\nu_n)_{n \geq 0})$ , write  $\bullet_A^{x_0, \dots, x_n}$  or  $\bullet_{K,A}^{x_0, \dots, x_n}$  for  $A \in \{X, Y\}$  to distinguish the composition operations in each of these spaces.

The ‘operadic’ flavor of  $\Phi_{(X, (\mu_n)_{n \geq 0})}^{x_0, \dots, x_n}$  is admittedly somewhat diluted in the following lemma concerning  $\Phi_f^{x_0, \dots, x_n}$ . Future work will include seeking out a perhaps more natural category than  $\mathbf{SCD}_n \times [1]$ , which appears too ‘coarse’ in the following results to lend itself to such an interpretation. Nonetheless, some operadic color still shines through, at least enough for us to prove what we need:

**Lemma 5.5.** *Let  $f : (X, (\mu_n)_{n \geq 0}) \rightarrow (Y, (\nu_n)_{n \geq 0})$  be a map in  $\mathbf{CSSP}_2^{comp}$ . Let  $n > 0$  and choose integers  $k_1, \dots, k_n \in \mathbb{Z}_{\geq 0}$  and a nested sequence of elements  $((x_j^i)_{j=0}^{k_i})_{i=1}^n$  of  $(X_{0,0})_0$  such that  $x_{k_i}^i = x_0^{i+1}$  for  $i < n$ . Let  $(x_0, \dots, x_r)$  be the flattened version of the sequence with all  $x_{k_i}^i$  removed for  $i < n$ . Set  $K$  to be the simplicial composition diagram*

$$K = (K_1 \sqcup_{\Delta[0]} \dots \sqcup_{\Delta[0]} K_n) \sqcup_{Sp(n)} K_0.$$

Then the diagram

$$\begin{array}{ccc} \bullet_{K,Y}^{f(x_0), \dots, f(x_r)} \circ h_1^c(f_{Sp(r)}^{x_0, \dots, x_r}) & & \\ \downarrow \Phi_f^{x_0, \dots, x_r}(1_K, 0 < 1) & \searrow \bullet_{K_0,Y}^{f(x_0^1), \dots, f(x_0^n), f(x_{k_n}^n)} \circ \prod_{i=1}^n \Phi_f^{x_0^i, \dots, x_{k_i}^i}(1_{K_i}, 0 < 1) & \\ & \bullet_{K_0,Y}^{f(x_0^1), \dots, f(x_0^n), f(x_{k_n}^n)} \circ h_1^c(f_{Sp(n)}^{x_0^1, \dots, x_0^n, x_{k_n}^n}) \circ \prod_{i=1}^n \bullet_{K_0,X}^{x_0^i, \dots, x_{k_i}^i} & \\ & \swarrow \Phi_f^{x_0^1, \dots, x_0^n, x_{k_n}^n}(1_{K_0}, 0 < 1) \circ \prod_{i=1}^n \bullet_{K_i,X}^{x_0^i, \dots, x_{k_i}^i} & \\ h_1^c(f_1^{x_0, \dots, x_r}) \circ \bullet_{K,X}^{x_0, \dots, x_r} & & \end{array}$$

commutes.

*Proof.* The proof is similar to that of Lemma 5.3. Expanding out all the terms, we obtain the expressions

$$\begin{aligned} R &:= h_1^c((\tau_K^*)^{f(x_0), f(x_r)}) \circ h_1^c(\eta_K^{f(x_0), f(x_r)}) \circ h_1^c(\mathcal{I}_{x_0, x_r}^{f, X}) \circ F_{x_0, \dots, x_r} \\ S &:= \bullet_{K_0, Y}^{f(x_0^1), \dots, f(x_0^n), f(x_{k_n}^n)} \circ \prod_{i=1}^n h_1^c((\tau_{K_i}^*)^{f(x_0^i), f(x_{k_i}^i)}) \circ h_1^c(\eta_{K_i}^{f(x_0^i), f(x_{k_i}^i)}) \circ h_1^c(\mathcal{I}_{x_0^i, x_{k_i}^i}^{f, X}) \circ F_{x_0^i, \dots, x_{k_i}^i} \\ T &:= h_1^c((\tau_{K_0}^*)^{f(x_0), f(x_r)}) \circ h_1^c(\eta_K^{f(x_0), f(x_r)}) \circ h_1^c(\mathcal{I}_{x_0, x_r}^{f, X}) \circ F_{x_0^1, \dots, x_0^n, x_{k_n}^n} \circ \prod_{i=1}^n \bullet_{K_0, Y}^{f(x_0^i), \dots, f(x_{k_i}^i)} \end{aligned}$$

If we can show both  $S$  and  $T$  to be of the form

$$h_1^c((\tau_K^*)^{f(x_0), f(x_r)}) \circ h_1^c(Q_A) \circ h_1^c(\mathcal{I}_{x_0, x_r}^{f, X}) \circ F_{x_0, \dots, x_r}$$

for some suitable induced left homotopies  $Q_A$  where  $A \in \{S, T\}$ , then we will be done.

Set  $D_k : X_{Sp(r)} \times N(I[1]) \rightarrow X_{Sp(r)} \times N(I[1])^k$  to the diagonal. For  $A \in \{X, Y\}$ , define the map

$$\mathcal{F}_{a_0, \dots, a_k}^A : \prod_{i=1}^k A(a_{i-1}, a_i) \rightarrow A_{Sp(k)}^{a_0, a_k}$$

to be the evident inclusion. Define  $I_{K,X}$  to be the identity homotopy from  $\mu_K$  to itself and  $I_{K,Y}$  to be as such for  $\nu_K$ .

Similarly to the proof of Lemma 5.3, we have that  $S = h_1^c((\tau_{K_0}^*)^{f(x_0),f(x_r)}) \circ h_1^c(\mathcal{S})$ , where  $\mathcal{S}$  is the map

$$(I_{K_0,Y}^{f(x_0),f(x_r)}) \circ \mathcal{F}_{f(x_0^1),\dots,f(x_0^n),f(x_{k_n}^n)}^Y \circ \left( \left( \prod_{i=1}^n Q_i \circ D_n^{x_0,\dots,x_r} \right) \times 1_{N(I[1])} \right) \circ D_2^{x_0,\dots,x_r}$$

where

$$Q_i := (\tau_{K_i}^* \circ \eta_{K_i})^{f(x_0^i),f(x_{k_i}^i)} \circ \mathcal{I}_{x_0^i,x_{k_i}^i}^{f,X_{Sp(k_i)} \times N(I[1])} \circ \mathcal{F}_{x_0^i,\dots,x_{k_i}^i}^X$$

and that  $T = h_1^c((\tau_{K_0}^*)^{f(x_0),f(x_r)}) \circ h_1^c(\mathcal{T})$ , where  $\mathcal{T}$  is the map

$$\eta_K^{f(x_0),f(x_r)} \circ \mathcal{I}_{x_0,x_r}^{f,X_{Sp(n)} \times N(I[1])} \circ \mathcal{F}_{x_0^1,\dots,x_0^n,x_{k_n}^n}^X \circ \left( \left( \prod_{i=1}^n W_i \circ D_n^{x_0,\dots,x_r} \right) \times 1_{N(I[1])} \right) \circ D_2^{x_0,\dots,x_r}$$

where

$$W_i := (\tau_{K_i}^*)^{x_0^i,x_{k_i}^i} \circ I_{K_i}^{x_0^i,x_{k_i}^i} \circ \mathcal{F}_{x_0^i,\dots,x_{k_i}^i}^X.$$

By similar manipulations to those in Lemma 5.3, we can reduce each of these further such that  $\mathcal{S} = Q_S^{f(x_0),f(x_r)} \circ \mathcal{I}_{x_0,x_r}^{f,X} \circ \mathcal{F}_{x_0,\dots,x_r}^X$  where

$$Q_S := I_{K_0,Y} \circ \left( \left( \prod_{i=1}^n (\tau_{K_i}^* \circ \eta_{K_i}) \circ D_n \right) \times 1_{N(I[1])} \right) \circ D_2$$

and such that  $\mathcal{T} = Q_T^{f(x_0),f(x_r)} \circ \mathcal{I}_{x_0,x_r}^{f,X} \circ \mathcal{F}_{x_0,\dots,x_r}^X$  where

$$Q_T := \eta_K \circ \left( \left( \prod_{i=1}^n (\tau_{K_i}^* \circ I_{K_i}) \circ D_n \right) \times 1_{N(I[1])} \right) \circ D_2.$$

Both of these are of suitable form for application of Lemma 5.2, the resulting homotopies from which are both of the form  $X_{Sp(r)} \times N(I[1]) \rightarrow Y_K$  and can be composed to complete the proof.  $\square$

Armed with these results, we are able to finally declare our definition of  $h_2(f)$  and prove its correctness.

**Definition 5.6.** Let  $f : (X, (\mu_n)_{n \geq 0}) \rightarrow (Y, (\nu_n)_{n \geq 0})$  be a morphism in  $\mathbf{CSSP}_2^{\text{comp}}$ . Then define

$$h_2(f) : h_2(X, (\mu_n)_{n \geq 0}) \rightarrow h_2(Y, (\nu_n)_{n \geq 0})$$

to be the pseudofunctor defined such that:

1. For every  $x \in (X_{0,0})_0$ ,  $h_2(f)(x) = f_{0,0}(x)$ ;
2. For every  $x, y \in (X_{0,0})_0$ ,  $h_2(f)_{x,y} = h_1(f_1^{x,y}) : h_1(X(x, y)) \rightarrow h_1(Y(x, y))$ ;
3. For every  $n \in \mathbb{N}$  and  $x_0, \dots, x_n \in (X_{0,0})_0$ , the natural isomorphism

$$\pi_{x_0,\dots,x_n} := \Phi_f^{x_0,\dots,x_n}(\Delta[n], 0 < 1);$$



4. For every  $x \in (X_{0,0})_0$ , the natural isomorphism  $\pi_x$  is the identity.

**Theorem 5.6.** Let  $f : (X, (\mu_n)_{n \geq 0}) \rightarrow (Y, (\nu_n)_{n \geq 0})$  be a morphism in  $\mathbf{CSSP}_2^{comp}$ . Then  $h_2(f)$  is a pseudofunctor.

*Proof.* Let  $n \in \mathbb{Z}_{>0}$  and  $k_1, \dots, k_n \in \mathbb{Z}_{\geq 0}$ , with  $r = \sum_{i=1}^n k_i$ . Consider any  $n$ -tuple of tuples of elements of  $(X_{0,0})_0$

$$Y := ((x_0^1, x_1^1, \dots, x_{k_1}^1), \dots, (x_0^n, \dots, x_{k_n}^n))$$

such that  $x_{k_i}^i = x_0^{i+1}$  for every  $i < n$ . Let  $(x_0, \dots, x_r)$  be the flattened tuple after removing each  $x_{k_i}^i$  for  $i < n$ .

Consider moreover the composition diagram

$$K := (\Delta[k_1] \sqcup_{\Delta[0]} \dots \sqcup_{\Delta[0]} \Delta[k_n]) \sqcup_{Sp(n)} \Delta[n].$$

We are tasked with showing that the diagram

$$\begin{array}{ccc} \bullet_{K,Y}^{f(x_0), \dots, f(x_r)} \circ h_1^c(f_{Sp(r)}^{x_0, \dots, x_r}) & \xrightarrow{\Phi_f^{x_0, \dots, x_r}(f_{k_1, \dots, k_n, 1_0})} & \bullet_Y^{x_0, \dots, x_r} \circ h_1^c(f_{Sp(r)}^{x_0, \dots, x_r}) \\ \downarrow \text{1}_{\bullet_Y}^{f(x_0^1), \dots, f(x_0^n), f(x_{k_n}^n)} \circ \prod_{i=1}^n \Phi_f^{x_0^i, \dots, x_{k_i}^i}(1_{K_i}, 0 < 1) & & \downarrow \Phi_f^{x_0, \dots, x_r}(1_{\Delta[r]}, 0 < 1) \\ \bullet_Y^{f(x_0^1), \dots, f(x_0^n), f(x_{k_n}^n)} \circ h_1^c(f_{Sp(n)}^{x_0^1, \dots, x_0^n, x_{k_n}^n}) \circ \prod_{i=1}^n \bullet_X^{x_0^i, \dots, x_{k_i}^i} & & \Phi_f^{x_0, \dots, x_r}(1_{\Delta[r]}, 0 < 1) \\ \downarrow \Phi_f^{x_0^1, \dots, x_0^n, x_{k_n}^n}(1_{K_0}, 0 < 1) \circ \prod_{i=1}^n \text{1}_{\bullet_{X, K_i}^{x_0^i, \dots, x_{k_i}^i}} & & \downarrow \\ h_1^c(f_1^{x_0, x_r}) \circ \bullet_{K,X}^{x_0, \dots, x_r} & \xrightarrow{\Phi_f^{x_0, \dots, x_r}(f_{k_1, \dots, k_n, 1_1})} & h_1^c(f_1^{x_0, x_r}) \circ \bullet_X^{x_0, \dots, x_r} \end{array}$$

commutes. This, however, must hold, by Lemma 5.5 and functoriality of  $\Phi_f^{x_0, \dots, x_r}$ . The other commuting diagram is trivial, as all involved natural isomorphisms are identities.  $\square$

We now need to show that  $h_2$ , given the behavior on objects from Definition 5.3 and morphisms from Definition 5.6, defines a valid functor into  $\mathbf{UBicat}$ :

**Theorem 5.7.** Let

$$(X, (\mu_n)_{n \geq 0}) \xrightarrow{f} (Y, (\nu_n)_{n \geq 0}) \xrightarrow{g} (Z, (\omega_n)_{n \geq 0})$$

be a chain of morphisms in  $\mathbf{CSSP}_2^{comp}$ . Then  $h_2(g \circ f) = h_2(g) \circ h_2(f)$ .

*Proof.* The equivalence is evident on the object mapping and behavior on hom-categories. For the non-trivial natural isomorphisms, let the isomorphisms for  $g$  be labelled as  $\theta$  and for  $f$  be  $\pi$ . The ones  $\psi$  for  $g \circ f$  must be such that

$$\psi_{x_0, \dots, x_n} = (h_1^c(g_1^{f(x_0), f(x_n)})(\pi_{x_0, \dots, x_n})) (\theta_{f(x_0), \dots, f(x_n)} \circ h_1^c(f_{Sp(n)}^{x_0, \dots, x_n})).$$

Expanding out all of these terms gives us that

$$\begin{aligned} \pi_{x_0, \dots, x_n} &= h_1^c(Y_{\langle 0, n \rangle}^{f(x_0), f(x_n)}) \circ h_1^c(\eta_n^{f(x_0), f(x_n)}) \circ h_1^c(\mathcal{I}_{x_0, x_n}^{f, X_{Sp(n)}}) \circ F_{x_0, \dots, x_n}^X \\ \theta_{f(x_0), \dots, f(x_n)} &= h_1^c(Z_{\langle 0, n \rangle}^{gf(x_0), gf(x_n)}) \circ h_1^c(\kappa_n^{gf(x_0), gf(x_n)}) \circ h_1^c(\mathcal{I}_{f(x_0), f(x_n)}^{g, Y_{Sp(n)}}) \circ F_{f(x_0), \dots, f(x_n)}^Y \\ \psi_{x_0, \dots, x_n} &= h_1^c(Z_{\langle 0, n \rangle}^{gf(x_0), gf(x_n)}) \circ h_1^c(\zeta_n^{gf(x_0), gf(x_n)}) \circ h_1^c(\mathcal{I}_{x_0, x_n}^{gf, X_{Sp(n)}}) \circ F_{x_0, \dots, x_n}^X \end{aligned}$$

where  $F_{a_0, \dots, a_n}^A$  is the map  $F_{a_0, \dots, a_n}$  adjusted in the obvious way for  $A \in \{X, Y\}$  and the left homotopies  $\eta_n, \kappa_n$  and  $\zeta_n$  are respectively from  $\nu_n \circ f_{Sp(n)}$  to  $f_n \circ \mu_n$ , from  $\omega_n \circ g_{Sp(n)}$  to  $g_n \circ \nu_n$  and from  $\omega_n \circ g_{Sp(n)} \circ f_{Sp(n)}$  to  $g_n \circ f_n \circ \mu_n$ .

Note that  $\theta_{f(x_0), \dots, f(x_n)} \circ h_1^c(f_{Sp(n)}^{x_0, \dots, x_n})$  is equal to

$$h_1^c(Z_{(0,n)}^{gf(x_0), gf(x_n)}) \circ h_1^c((\kappa_n \circ (f_{Sp(n)} \times 1_{N(I[1])}))^{gf(x_0), gf(x_n)}) \circ h_1^c(\mathcal{I}_{x_0, x_n}^{gf, X_{Sp(n)}}) \circ F_{x_0, \dots, x_n}^X$$

while similarly  $h_1^c(g_1^{f(x_0), f(x_n)})(\pi_{x_0, \dots, x_n})$  is equal to

$$h_1^c(g_1^{f(x_0), f(x_n)}) \circ h_1^c(Y_{(0,n)}^{f(x_0), f(x_n)}) \circ h_1^c(\eta_n^{f(x_0), f(x_n)}) \circ h_1^c(\mathcal{I}_{x_0, x_n}^{f, X_{Sp(n)}}) \circ F_{x_0, \dots, x_n}^X$$

which is then equal to

$$h_1^c(Z_{(0,n)}^{gf(x_0), gf(x_n)}) \circ h_1^c((g_n \circ \eta_n)^{f(x_0), f(x_n)}) \circ h_1^c(\mathcal{I}_{x_0, x_n}^{f, X_{Sp(n)}}) \circ F_{x_0, \dots, x_n}^X.$$

All of these three cases are now of the form  $h_1^c(Z_{(0,n)}^{gf(x_0), gf(x_n)}) \circ h_1^c(\Gamma^{f(x_0), f(x_n)}) \circ h_1^c(\mathcal{I}_{x_0, x_n}^{f, X_{Sp(n)}}) \circ F_{x_0, \dots, x_n}^X$  for left homotopies  $\Gamma$ . We find that the left homotopies  $g_n \circ \eta_n$  and  $\kappa_n \circ (f_{Sp(n)} \times 1_{N(I[1])})$  are now readily composable, so by Proposition 4.13 and Corollary 4.6.1 the equality holds.  $\square$

**Theorem 5.8.** *Let  $(X, (\mu_n)_{n \geq 0})$  be an object in  $\mathbf{CSSP}_2^{comp}$ . Then  $h_2(1_X) = 1_{h_2(X, (\mu_n)_{n \geq 0})}$ .*

*Proof.* The behavior on objects and hom-categories is self-evident. Moreover,

$$\Phi_{1_X}^{x_0, \dots, x_n}(1_{\Delta[n]}, 0 < 1)$$

is built of a left homotopy with the same domain and codomain, so can be assumed to be constant, making the natural isomorphisms trivial.  $\square$

We are finally able to state what we consider to be the central definition of this paper:

**Definition 5.7.** *Define*

$$h_2 : \mathbf{CSSP}_2^{comp} \rightarrow \mathbf{UBicat}$$

*to be the functor that sends  $(X, (\mu_n)_{n \geq 0})$  to  $h_2(X, (\mu_n)_{n \geq 0})$  and a morphism  $f$  to  $h_2(f)$ .*

A useful consequence of functoriality is that we may finally show how all choices of horizontal compositions made in a complete 2-fold Segal space  $X$  ultimately have no real consequence in the homotopy bicategory:

**Corollary 5.8.1.** *Let  $X$  be a complete 2-fold Segal space. Let  $(\mu_n)_{n \geq 0}$  and  $(\nu_n)_{n \geq 0}$  be two choices of horizontal compositions for  $X$ . Then there is a pseudofunctor  $h_2(X, (\mu_n)_{n \geq 0}) \rightarrow h_2(X, (\nu_n)_{n \geq 0})$  which acts as the identity on objects and hom-categories.*

## 5.4 Fundamental Bigroupoids

We finally take stock of our progress by applying our construction to the case of  $X = S := \mathbf{Sing}_{\text{SSS}}(U)$  for some given  $U \in \mathbf{Top}$ . We will find this yields a sensible notion of *fundamental bigroupoid* of a topological space, similar in nature to [22] though with differences in horizontal composition.

To begin, we have a set of objects  $\mathbf{ob}(h_2(S))$  equal to the set of points in  $U$ , along with hom-categories of the form

$$\mathbf{Hom}_{h_2(S)}(x, y) \cong \Pi_1(\{(x, y)\} \times_{U \times U} U^{\Delta_t[1]})$$

which is precisely what we should expect.

For composition, we observe that  $\mathbf{Sing}_{\mathbf{SS}}$  commutes with limits, since it is a right adjoint. Hence,

$$\prod_{S_0}^{1 \leq i \leq n} S_1 \cong \mathbf{Sing}_{\mathbf{SS}}(U^{Sp_t(n)})$$

where  $Sp_t(n) := \Delta_t[1] \sqcup_{\Delta_t[0]} \cdots \sqcup_{\Delta_t[0]} \Delta_t[1]$ . Thus, a choice of horizontal compositions for  $S$  actually reduces to solving lifting problems of the form

$$\begin{array}{ccc} & \mathbf{Sing}_{\mathbf{SS}}(U^{\Delta_t[n]}) & \\ & \nearrow \mu_n \text{ (dashed)} & \downarrow \mathbf{Sing}_{\mathbf{SS}}(U^{|g_n|}) \\ \mathbf{Sing}_{\mathbf{SS}}(U^{Sp_t(n)}) & \xrightarrow{id} & \mathbf{Sing}_{\mathbf{SS}}(U^{Sp_t(n)}) \end{array}$$

where  $|g_n| : Sp_t(n) \hookrightarrow \Delta_t[n]$  is the topological spine inclusion.

Note that the map  $|g_n|$  is a trivial Hurewicz cofibration; indeed it is the inclusion of a subcomplex and is moreover a homotopy equivalence. This in turn implies that  $U^{\Delta_t[n]} \rightarrow U^{Sp_t(n)}$  is a trivial Hurewicz fibration. Hence, solutions to this lifting problem can be obtained from any of the sections of  $U^{|g_n|}$ . We can obtain a simple example of such a section by finding a deformation retract  $f_n : \Delta_t[n] \rightarrow Sp_t[n]$  of  $|g_n|$ : if we understand that  $Sp_t(n) \subseteq \Delta_t[n]$  and define  $e_i \in \Delta_t[n]$  such that  $e_i$ 's  $i^{\text{th}}$  entry is 1, we choose to map  $e_i \mapsto (\frac{n-i}{n}, 0, \dots, 0, \frac{i}{n})$ , inducing linearly a retraction  $r_n$  to the edge from  $e_0$  to  $e_n$  in  $\Delta_t[n]$  that sends

$$(x_0, \dots, x_n) \mapsto \left( \sum_{i=0}^n x_i \frac{n-i}{n}, 0, \dots, 0, \sum_{i=0}^n x_i \frac{i}{n} \right).$$

Composing this with a map  $s_n$  to  $Sp_t(n)$  that sends

$$(1 - x_n, 0, \dots, 0, x_n) \mapsto ((i+1) - nx_n)e_i + (nx_n - i)e_{i+1}, \quad \frac{i}{n} \leq x_n \leq \frac{i+1}{n}$$

gives us a final retraction  $R_n = s_n \circ r_n : \Delta_t[n] \rightarrow Sp_t(n)$ , sending

$$(x_0, \dots, x_n) \mapsto \left( (i+1) - \sum_{j=0}^n jx_j \right) e_i + \left( \sum_{j=0}^n jx_j - i \right) e_{i+1}, \quad i \leq \sum_{j=0}^n jx_j \leq i+1$$

We should show that this is the identity on  $Sp_t(n)$ . Indeed, consider some  $(x_0, \dots, x_n) = (1-t)e_i + te_{i+1} = (0, \dots, 0, 1-t, t, 0, \dots, 0)$  for  $0 \leq t \leq 1$ . Then

$$\sum_{j=0}^n jx_j = i(1-t) + (i+1)t = i - it + it + t = i + t$$

which ranges from  $i$  to  $i+1$  with  $t$ .

Applying  $R_n$  then gives us

$$R_n(0, \dots, 0, 1-t, t, 0, \dots, 0) = ((i+1) - (i+t))e_i + (i+t - i)e_{i+1} = (1-t)e_i + te_{i+1}$$

which confirms that the inclusion  $|g_n| : Sp_t(n) \subseteq \Delta[n]$  is a section of  $R_n$ , as needed.

We now have our composition map, which by using the identification

$$\mathbf{Hom}_{h_2(S)}(x_0, x_1) \times \cdots \times \mathbf{Hom}_{h_2(S)}(x_{n-1}, x_n) \cong \Pi_1(\{x_0, \cdots, x_n\} \times_{U^{n+1}} U^{Sp_t(n)})$$

can be phrased in the form

$$\begin{array}{ccc} \Pi_1(\{x_0, \cdots, x_n\} \times_{U^{n+1}} U^{Sp_t(n)}) & & \\ \downarrow & & \\ \Pi_1(\{x_0, x_n\} \times_{U \times U} U^{Sp_t(n)}) & \xrightarrow{\Pi_1(1_{\{x_0, x_n\}} \times_{1_{U^2}} U^{R_n})} & \Pi_1(\{x_0, x_n\} \times_{U \times U} U^{\Delta_t[n]}) \\ & & \downarrow \\ & & \Pi_1(\{x_0, x_n\} \times_{U \times U} U^{\Delta_t[1]}) \end{array}$$

which, in the end, amounts to composing a sequence of  $n$  paths in a topological space into one path. This composition, by the way  $R_n$  was defined, is ‘unbiased’ - the composite path  $[0, 1] \rightarrow U$  sends the interval  $[\frac{i}{n}, \frac{i+1}{n}]$  to the  $i^{\text{th}}$  path in the chain by the map  $x \mapsto nx - i$ .

We should note that this is but one possible choice of composition operation. In our approach, we established that choice of composition could be built upon a choice of deformation retract for the inclusion  $Sp_t(n) \hookrightarrow \Delta_t[n]$ . There are in fact an uncountably infinite number of such retracts. For instance, if  $n = 2$ , choosing any  $t \in (0, 1)$  induces a distinct retract  $\Delta_t[2] \rightarrow Sp_t(2)$  sending

$$(a, b, c) \mapsto \begin{cases} (a, 1 - a, 0) & bt + c < t \\ (0, 1 - c, c) & bt + c \geq t. \end{cases}$$

This results in a composition of paths which, as before, concatenates the paths, but differs in the parameterization of the concatenation. We can generalize this phenomenon to all  $\Delta_t[n]$  in a straightforward way.

Of course, the space of all deformation retracts  $\Delta_t[n] \rightarrow Sp_t[n]$  need not contain only piecewise linear elements or even only smooth ones. Moreover, our particular construction of the homotopy bicategory of  $S$  may have used these deformation retracts, but there is certainly no need to rely on these to obtain the necessary lifts at all. Any section of  $U|g_n| : U^{\Delta_t[n]} \rightarrow U^{Sp_t(n)}$  will do. One could go further and say the section need not result from a map on these underlying topological spaces - it need only be defined on the resulting complete Segal spaces, so behavior levelwise could potentially vary drastically.

We now turn to associators in  $h_2(S)$ , as unitors are of course trivial. In order to work within **Top** efficiently, we need to show that one may convert from homotopies defined using the interval  $[0, 1]$  to ones defined using  $N(I[1])$ .

**Proposition 5.5.** *There is a weak equivalence of complete Segal spaces  $\tau : N(I[1]) \rightarrow \mathbf{Sing}_{sS}([0, 1])$  such that the diagram*

$$\begin{array}{ccc} * \sqcup * & \longrightarrow & \mathbf{Sing}_{sS}([0, 1]) \\ \downarrow & \nearrow s & \downarrow \\ N(I[1]) & \longrightarrow & * \end{array}$$

*commutes.*

*Proof.* Note that this is a valid lifting problem, since  $\mathbf{Sing}_{\mathbf{sSet}}(-)$  preserves weak equivalences levelwise and has its image in  $\mathbf{CSSP}$ . Thus, the map  $S$  is simply a solution of a lifting problem, so will exist. Moreover, it must be a weak equivalence by 2-out-of-3.

One can concretely define  $S$  levelwise. Let  $n, m \geq 0$ . Recall that

$$N(I[1])_{n,m} \cong \mathbf{Hom}_{\mathbf{Cat}}([n] \times [m], I[1])$$

Note that  $[n] \times [m] \cong \mathbf{Ho}(\Delta[n] \times \Delta[m])$ , where  $\mathbf{Ho} : \mathbf{sSet} \rightarrow \mathbf{Cat}$  is the left adjoint to  $\mathbf{nerve} : \mathbf{Cat} \rightarrow \mathbf{sSet}$ . Thus, we have that

$$N(I[1])_{n,m} \cong \mathbf{Hom}_{\mathbf{sSet}}(\Delta[n] \times \Delta[m], \mathbf{nerve}(I[1])).$$

Applying  $|\cdot| : \mathbf{sSet} \rightarrow \mathbf{Top}$  then gives a map

$$N(I[1])_{n,m} \rightarrow \mathbf{Hom}_{\mathbf{Top}}(\Delta_t[n] \times \Delta_t[m], |\mathbf{nerve}(I[1])|).$$

This induces a map  $N(I[1]) \rightarrow \mathbf{Sing}_{\mathbf{sSet}}(|\mathbf{nerve}(I[1])|)$  in  $\mathbf{CSSP}$ <sup>8</sup>. Now, we seek a map  $|\mathbf{nerve}(I[1])| \rightarrow [0, 1]$  in  $\mathbf{Top}$ , as postcomposition with this will yield our desired morphism. By adjunction, it will suffice to find a map  $q : \mathbf{nerve}(I[1]) \rightarrow \mathbf{Sing}([0, 1])$ .

This map can be defined levelwise.  $q_0$  is a map from a two-element set; we send one element to 0 and the other to 1. The two nondegenerate 1-simplices are then sent by  $q_1$  to the maps  $\Delta_t[1] \rightarrow [0, 1]$  sending  $t \mapsto t$  and  $t \mapsto 1 - t$  respectively. This clearly respects degeneracy and face maps at level 0 and 1. Now, since  $\mathbf{nerve}(I[1])$  is a nerve, the maps

$$\mathbf{nerve}(I[1])_n \rightarrow \mathbf{nerve}(I[1])_1 \times_{\mathbf{nerve}(I[1])_0} \cdots \times_{\mathbf{nerve}(I[1])_0} \mathbf{nerve}(I[1])_1$$

are bijections for all  $n \geq 2$ , so every such  $q_n$  is uniquely induced in a way commuting immediately with face and degeneracy maps. This completes the construction of  $q$  and therefore  $S$ . Since  $S$  solves the above lifting problem, by 2-out-of-3  $S$  is again a weak equivalence.  $\square$

Another way to understand  $q$  is to notice that the elements of  $\mathbf{nerve}(I[1])_n$  can be identified with sequences  $(w_i)_{0 \leq i \leq n}$  where  $w_i \in \{0, 1\}$  for all  $i$ , with face and degeneracy maps given by copying or removing elements of the sequence. Indeed, an element of  $\mathbf{nerve}(I[1])_n$  is a chain of morphisms of length  $n$  in  $I[1]$ , which is uniquely identified by the sequence of objects in the chain. The image of such a sequence is then the unique affine map  $\Delta_t[n] \rightarrow [0, 1]$  sending  $e_i \mapsto w_i$ . This is precisely the notion of a *thick 1-simplex* given by Getzler in [17], who notes this has previously been called  $E(1)$  by Rezk in [42] and  $\Delta'[1]$  by Joyal and Tierney in [29].

This means, given we specify our homotopies classically in  $\mathbf{Top}$ , we can transmit them by  $\mathbf{Sing}_{\mathbf{sSet}}$  and precomposition with  $S$  to the format of left homotopy we have built our work upon.

Suppose then we have chosen sections  $p_n : U^{Sp_t(n)} \rightarrow U^{\Delta_t[n]}$  of  $U^{|g_n|}$ . It will suffice to find a homotopy

$$U^{Sp_t(r)} \times [0, 1] \rightarrow U^{(\Delta_t[k_1] \sqcup_{\Delta_t[0]} \cdots \sqcup_{\Delta_t[0]} \Delta_t[k_n]) \sqcup_{Sp_t(n)} \Delta_t[n]}$$

from  $U^{|f_{k_1, \dots, k_n}|} \circ p_n$  to  $(p_{k_1} \times_{1_U} \cdots \times_{1_U} p_{k_n}) \times_{U^g} p_n$  that is constant on vertices, where  $g : Sp_t(n) \rightarrow Sp_t(r)$  is the piecewise linear map sending  $e_i \mapsto e_{\sum_{j=1}^i k_j}$ . Both of these maps

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<sup>8</sup>Note that this is not levelwise a bijection.

are again sections of a trivial Hurewicz fibration, so there will necessarily be a homotopy between them.

In the simple case that  $p_n := U^{R_n}$ , our challenge shrinks to the problem of finding a homotopy from  $U^{|f_{k_1, \dots, k_n}|} \circ U^{R_r}$  to  $U^{(R_{k_1} \sqcup_{id} \dots \sqcup_{id} R_{k_n}) \sqcup_g R_n}$ . It then suffices to construct a homotopy of the form

$$\left( (\Delta_t[k_1] \sqcup_{\Delta_t[0]} \dots \sqcup_{\Delta_t[0]} \Delta_t[k_n]) \sqcup_{Sp_t(n)} \Delta_t[n] \right) \times [0, 1] \rightarrow Sp_t(r)$$

from  $R_r \circ |f_{k_1, \dots, k_n}|$  to  $(R_{k_1} \sqcup_{id} \dots \sqcup_{id} R_{k_n}) \sqcup_g R_n$ . This is of course possible, as the target space is contractible, so any two maps into it are homotopic to any constant map and thus each other. We choose to identify  $Sp_t(r)$  with  $[0, r]$  in the evident way, sending  $e_i \mapsto \frac{i}{r}$ , and set our homotopy to be linear interpolation.

To understand more concretely what our homotopy does, consider the case  $n = 2$  and  $k_1 = 1, k_2 = 2$ . The homotopy is from the composition map  $(- \circ (- \circ -))$  to  $(- \circ - \circ -)$ . We have reduced this to a homotopy through deformation retracts

$$\left( (\Delta_t[1] \sqcup_{\Delta_t[0]} \Delta_t[2]) \sqcup_{Sp_t(2)} \Delta_t[2] \right) \times [0, 1] \rightarrow [0, 3] \cong Sp_t(3).$$

The important behavior of the homotopy is on the unit square  $\Delta_t[1] \times [0, 1]$  in the domain, identified by the inclusion

$$\Delta_t[1] \xrightarrow{|(0,2)|} \Delta_t[2] \cong (\emptyset \sqcup_{\emptyset} \emptyset) \sqcup_{\emptyset} \Delta_t[2] \hookrightarrow \left( (\Delta_t[1] \sqcup_{\Delta_t[0]} \Delta_t[2]) \sqcup_{Sp_t(2)} \Delta_t[2] \right)$$

as the image of this path is the end result of the composition operation. We find that  $R_3 \circ |f_{1,2}|$  acts on this interval as the morphism  $[0, 1] \rightarrow [0, 3]$  sending  $x \mapsto 3x$ , while  $(R_1 \sqcup_{id} R_2) \sqcup_g R_2$  is the piecewise linear map sending  $0 \mapsto 0, \frac{1}{2} \mapsto 1, \frac{3}{4} \mapsto 2$  and  $1 \mapsto 3$ . Our associator is a piecewise linear interpolation between these two maps.

Once again, there are many possible choices of homotopy here. Linear interpolation is but one interpolation possible - they may be only polynomial, smooth or merely continuous. However, all of these will produce identical associators, as the induced natural isomorphisms will levelwise be homotopic paths. In order to general truly distinct associators, they must be given as natural isomorphisms that are levelwise paths which are not homotopic, which is not possible by the given approach nor indeed by any method within the confines of our construction.

Now we consider the production of pseudofunctors. We add the following definition:

**Definition 5.8.** Let  $\mathbf{Top}^{comp}$  be the category of pairs  $(T, (p_n)_{n \geq 0})$  where  $T \in \mathbf{Top}$  and  $p_n : T^{Sp_t(n)} \rightarrow T^{\Delta_t[n]}$  is a section of the map  $T^{|g_n|}$ , together with maps  $(T, p_n) \rightarrow (U, q_n)$  given by continuous maps  $T \rightarrow U$  in  $\mathbf{Top}$ .

We now have a functor

$$\mathbf{Sing}_{\text{sss}}^{comp} : \mathbf{Top}^{comp} \rightarrow \mathbf{CSSP}_2^{comp}$$

that extends  $\mathbf{Sing}_{\text{sss}}$  by inducing horizontal compositions from the maps  $p_n$ .

Consider a continuous map  $f : (T, p_n) \rightarrow (U, q_n)$  in  $\mathbf{Top}$ . Let  $S_T := \mathbf{Sing}_{\text{sss}}(T)$  and  $S_U := \mathbf{Sing}_{\text{sss}}(U)$ . We already have specified horizontal compositions in  $S_U$  and  $S_T$ . Write  $\mu_n$  for the horizontal compositions induced by  $p_n$  and likewise  $\nu_n$  for those from  $q_n$ . We thus obtain a map

$$F := \mathbf{Sing}_{\text{sss}}(f) : (S_T, \mu_n) \rightarrow (S_U, \nu_n).$$

Write  $F$  for the map  $S_T \rightarrow S_U$  as well.

The pseudofunctor  $h_2(F) : h_2(S_T, (\mu_n)_{n \geq 0}) \rightarrow h_2(S_U, (\nu_n)_{n \geq 0})$  evidently sends  $x \mapsto f(x)$  for objects  $x \in T$ . On hom-categories, for  $x, y \in T$ , we have the induced functor

$$\Pi_1(\{(x, y)\} \times f^{\Delta_t[1]}) : \Pi_1(\{(x, y)\} \times T^{\Delta_t[1]}) \rightarrow \Pi_1(\{(f(x), f(y))\} \times U^{\Delta_t[1]})$$

which sends a 1-morphism  $p : [0, 1] \rightarrow T$  from  $x$  to  $y$  to the path  $f \circ p$ . Moreover, it sends a homotopy class of paths  $[H]$  for  $H : [0, 1] \times [0, 1] \rightarrow T$  to  $[f \circ H]$ .

Vertical composition is respected on the nose. For horizontal composition, we must understand the isomorphisms  $\pi_{x_0, \dots, x_n}$ . It suffices to find a homotopy

$$T^{Sp_t(n)} \times [0, 1] \rightarrow U^{\Delta_t[n]}$$

from  $f^{\Delta_t[n]} \circ p_n$  to  $q_n \circ f^{Sp_t(n)}$  that is constant on the vertices. Such a homotopy will necessarily exist, as both are solutions to the same lifting problem. Any two such homotopies will by Corollary 5.8.1 induce the same natural isomorphism, so the pseudofunctor  $h_2(F)$  is specified entirely.

In the special case that  $p_n = T^{R_n}$  and  $q_n = U^{R_n}$ , this is much simpler. In fact, the two maps being homotoped between are equal, so the natural isomorphisms  $\pi_{x_0, \dots, x_n}$  are all identities.

Now, suppose one introduced a different set of deformation retracts  $R'_n : \Delta[n] \rightarrow Sp_t(n)$ . These induce a different choice of horizontal compositions  $(\eta_n)_{n \geq 0}$  on  $S_T$ . We thus have by Corollary 5.8.1 that there is a canonical pseudofunctor  $P : h_2(S_T, (\mu_n)_{n \geq 0}) \rightarrow h_2(S_T, (\eta_n)_{n \geq 0})$  that is the identity on objects and hom-categories. Indeed,  $P$  is induced by the identity morphism  $1_T : T \rightarrow T$ .

The only interesting aspect of this pseudofunctor's structure is in the natural isomorphisms  $\pi_{x_0, \dots, x_n}$ . These will be induced by homotopies

$$T^{Sp_t(n)} \times [0, 1] \rightarrow T^{\Delta_t[n]}$$

from  $U^{R_n}$  to  $U^{R'_n}$ . For such purposes, it suffices to find a left homotopy

$$\Delta_t[n] \times [0, 1] \rightarrow Sp_t(n)$$

from  $R_n$  to  $R'_n$ . One must of course exist, as the codomain is contractible. We may choose our left homotopy to be linear interpolation pointwise, choosing to interpret  $Sp_t(n) \cong [0, n]$ . We find thus that, for a sequence of paths  $x_0 \xrightarrow{f_1} x_1 \cdots \xrightarrow{f_n} x_n$ , the natural isomorphism  $\pi_{(f_1, \dots, f_n)}$  is the homotopy class  $[H]$  of the map  $H : [0, 1] \rightarrow U^{[0, 1]}$  such  $0 \mapsto R_n \circ (f_0 \sqcup_* \cdots \sqcup_* f_n)$  and  $1 \mapsto R'_n \circ (f_0 \sqcup_* \cdots \sqcup_* f_n)$ , with all other points defined by the linear interpolation between  $R_n$  and  $R'_n$ . Any other homotopy will yield the same natural isomorphism, so this description covers all possibilities.

In the end, composing  $\mathbf{Sing}_{\text{SSS}}^{\text{comp}}$  and  $h_2$  yields a 'fundamental bigroupoid' functor

$$\Pi_2^{\text{comp}} : \mathbf{Top}^{\text{comp}} \rightarrow \mathbf{UBicat}.$$

Using some coherent choice of maps  $p_n$ , for instance those induced by  $R_n$ , allows us to make the domain  $\mathbf{Top}$ .

**Data Access Statement:** No data was collected or used in this project. All commutative diagrams were generated with <https://q.uiver.app/>.

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